Random walk on the incipient infinite cluster for oriented percolation in high dimensions

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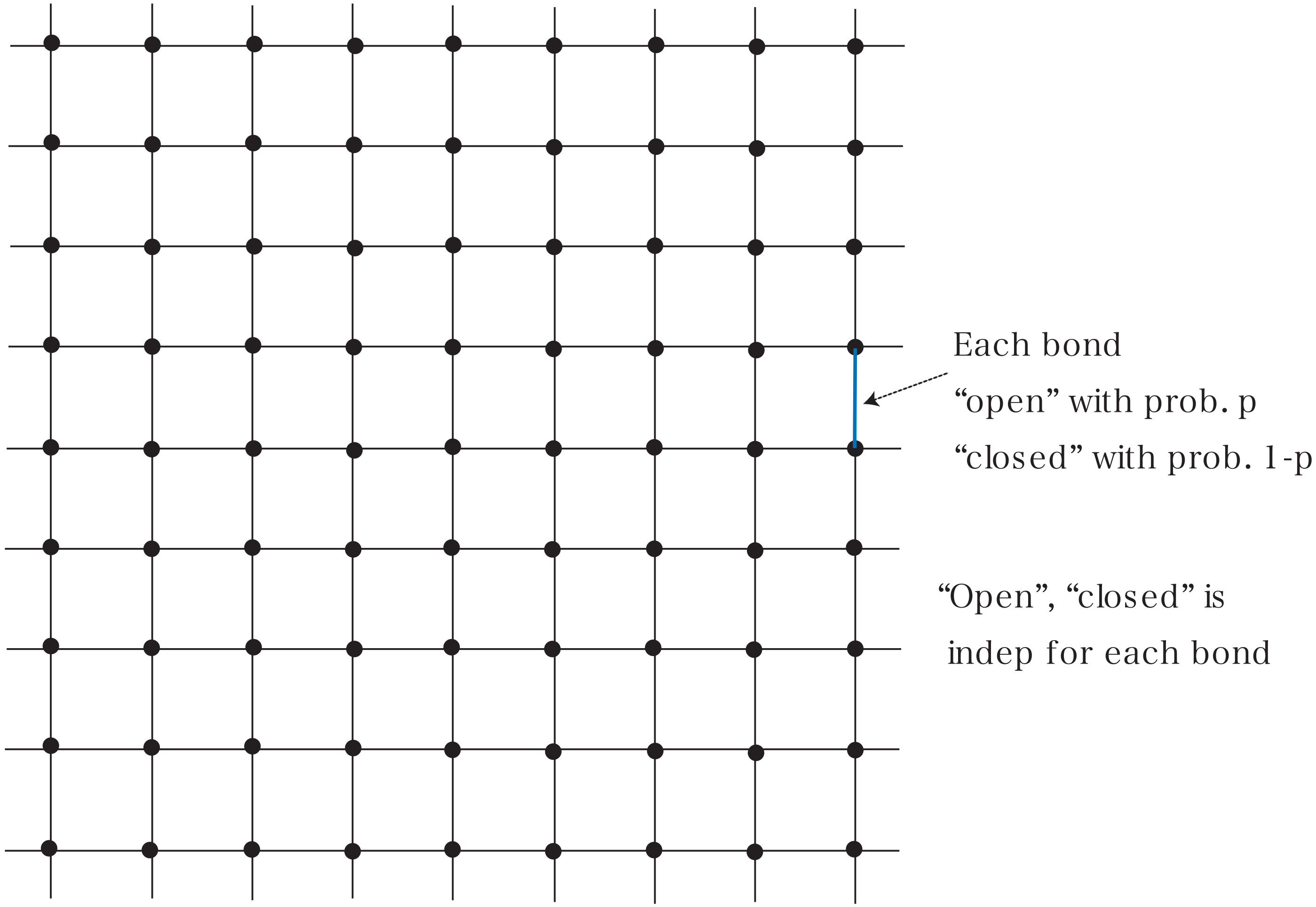
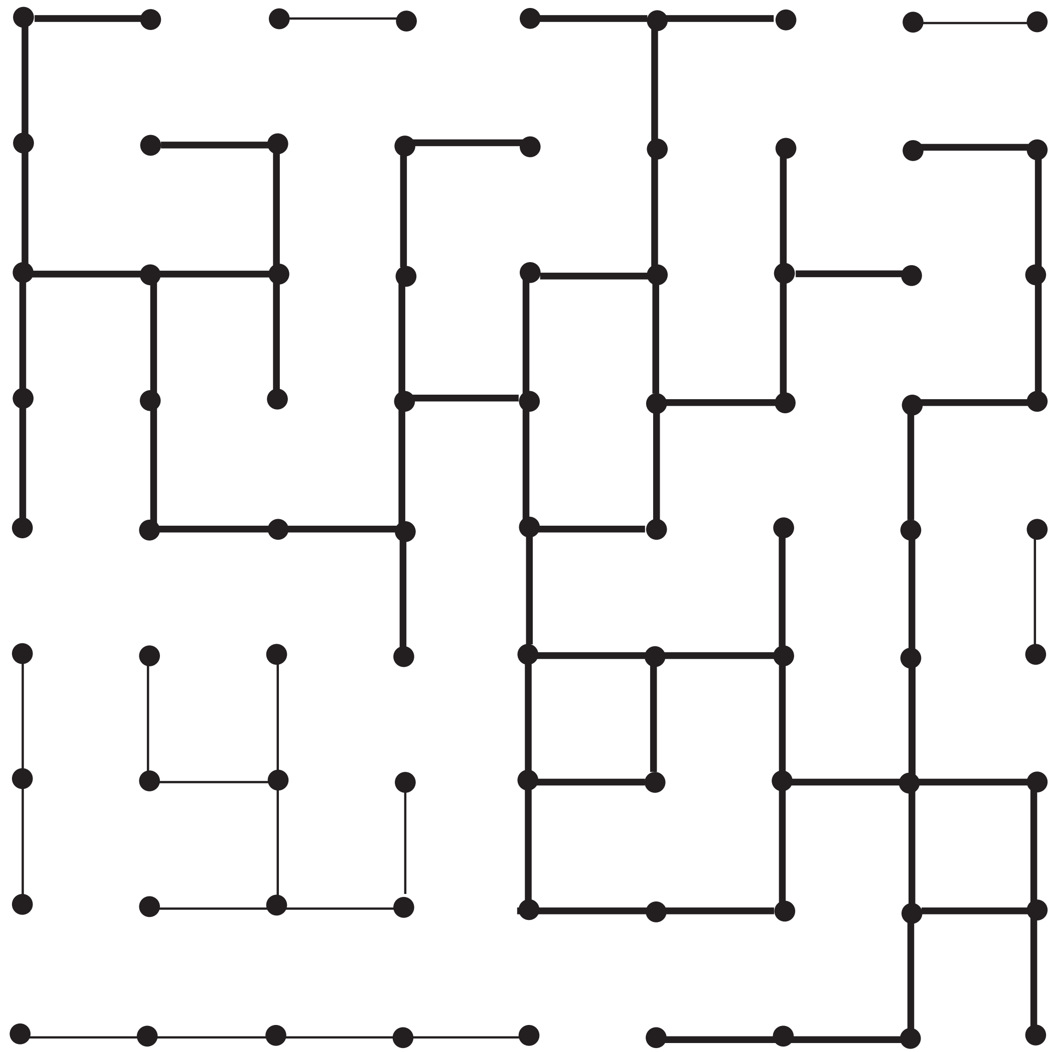
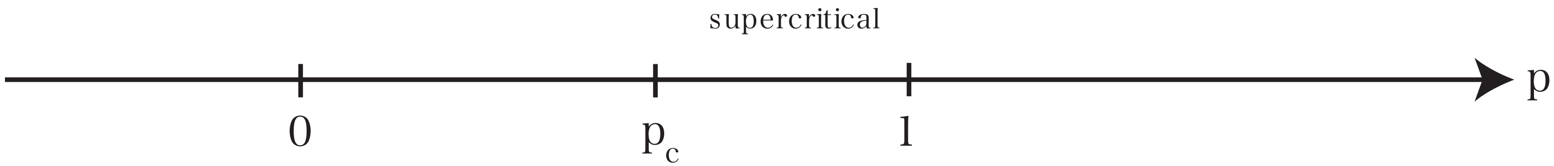
1 RW on the percolation cluster on $\mathbb{Z}^d$ $(d \geq 2)$





$\exists\ p_c\ :\ p \leq p_c$ --> no infinite cluster

$p > p_c$ --> unique infinite cluster



Supercritical case ($p > p_c$) Various results proved quite recently.

Mathieu-Remy ('04), Barlow ('04), Sidoravicius-Sznitman ('04), Berger-Biskup etc.

$$\frac{1}{\sqrt{n}} Y^{\omega}_{[n\,\cdot]} \xrightarrow{n\to\infty} B_{\sigma\,\cdot}, \qquad\qquad \mathbb{P}-\text{a.e. } \omega.$$

Critical case ($p = p_c$) Almost no mathematically rigorous results!!

Math. Physicists' approach (Ben-Avraham and S. Havlin (1987, Book: 2000))

'Anomalous' behaviour of the random walk

Let $d_s = -2\lim_{n\to\infty} \log p_{2n}(x,x)/\log n$.

Alexander-Orbach conjecture (J. Phys. Lett., 1982) $d \geq 2 \Rightarrow d_s = 4/3$.

Kesten (1986): $d = 2$ 'subdiffusive behaviour', i.e. $\exists \epsilon > 0$ s.t. $n^{-\frac{1}{2}+\epsilon} d(0, Y_n)$ is tight.

(Related work) $d = 2$: Smirnov, Lawler, Schramn, Werner etc. $\Rightarrow$ Shape of cont. limit

2 RW for critical oriented percolation cluster in $\mathbb{Z}^d$ $(d > 6)$

Original graph: vertices $\mathbb{Z}^d \times \mathbb{Z}_+$,

oriented bonds $((x, n), (y, n+1))$, $n \geq 0$, $x, y \in \mathbb{Z}^d$ with $0 \leq \|x - y\|_\infty \leq \exists L$.

We consider bond percolation on the graph.

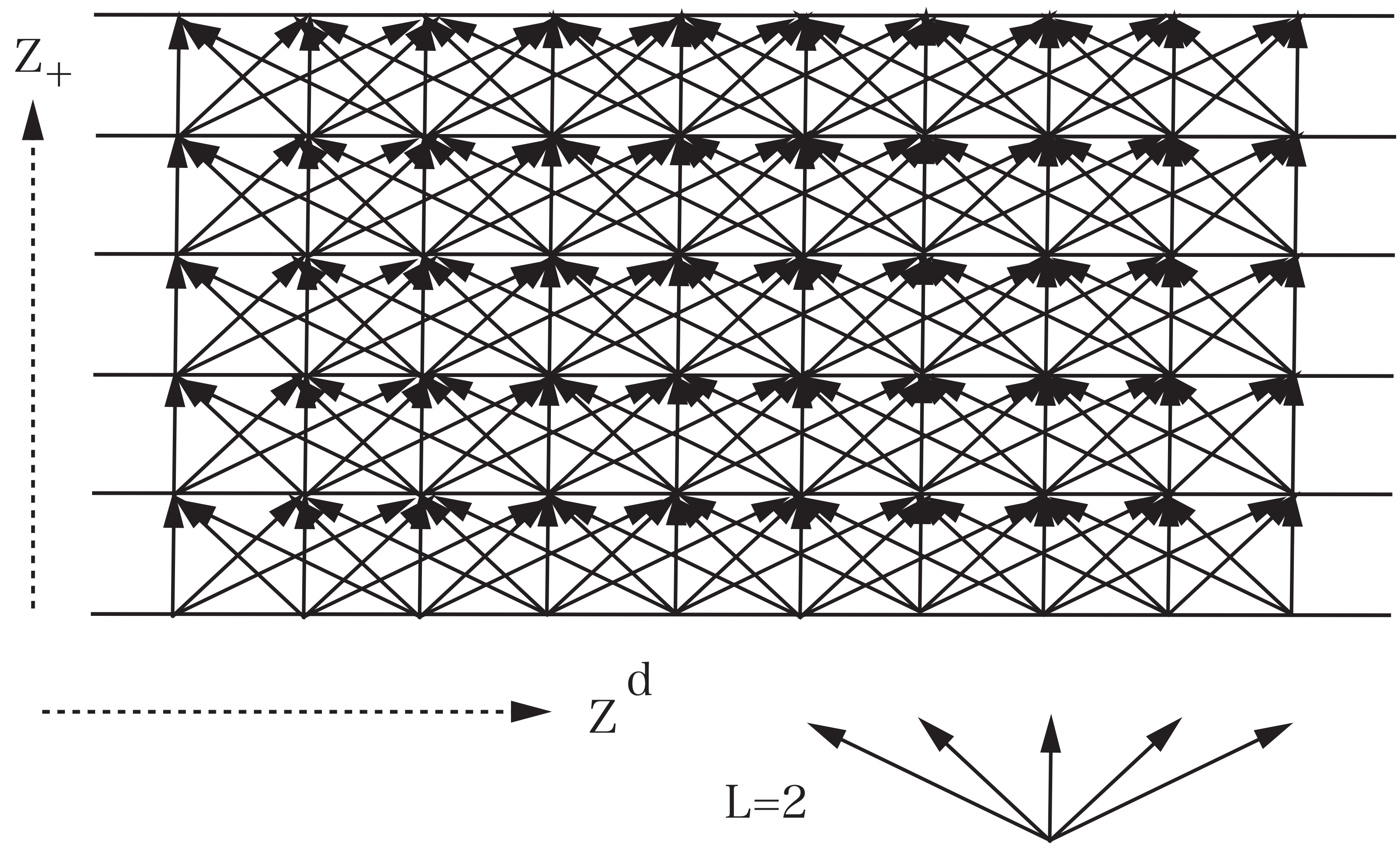
$$\mathcal{C}(x, n) = \{(y, m) : (x, n) \to (y, m)\}$$

$\exists p_c = p_c(d, L) \in (0, 1)$ s.t. $p > p_c \Rightarrow \exists$ infinite cluster, $p \leq p_c \Rightarrow$ no infinite cluster

So, at $p = p_c$, $\mathcal{C}(0, 0)$ is a finite cluster with prob. 1!

$\Rightarrow$ Consider Incipient infinite cluster (IIC). (I.e. at the critical prob., conditioned on extending to infinity) $\exists L_0(d)$ s.t. $L \geq L_0(d) \Rightarrow$ the existence of IIC is guaranteed.

(van der Hofstad, Hollander and G. Slade '02)



Z+
Z d
L=2

$(\Omega, \mathcal{F}, \mathbb{P})$: prob. space for the randomness of the space, $(\mathcal{G}(\omega), \omega \in \Omega)$: IIC

For each $\mathcal{G} = \mathcal{G}(\omega)$, let $\{Y_n\}$ be a simple RW on $\mathcal{G}$.

P_ω^x: law of $\{Y_n\}$ starting at $x \in \mathcal{G}(\omega)$

E_ω^x: its average, $p_n^\omega(x,y) := \mathbb{P}^x(Y_n = y)/\mu_y$.

Theorem 2.1 *$\exists \alpha_1, \alpha_2, \alpha_3 < \infty$, $\exists \Omega_0 \subset \Omega$ with $\mathbb{P}(\Omega_0) = 1$ s.t. the following holds.*

(a) $\forall \omega \in \Omega_0$ and $\forall x \in \mathcal{G}(\omega)$, $\exists N_x(\omega) < \infty$ s.t.

$$(\log n)^{-\alpha_1} n^{-2/3} \le p_{2n}^\omega(x,x) \le (\log n)^{\alpha_1} n^{-2/3}, \quad \forall n \ge N_x(\omega).$$

Especially, $d_s(G(\omega)) = \frac{4}{3}$, $\mathbb{P}$–a.s. ω (solves the A-O conj.), and the RW is recurrent.

(b) $\forall \omega \in \Omega_0$ and $\forall x \in \mathcal{G}(\omega)$, $\exists R_x(\omega) < \infty$ s.t.

$$(\log R)^{-\alpha_2} R^3 \le E_\omega^x \tau_R \le (\log R)^{\alpha_2} R^3, \quad \forall R \ge R_x(\omega),$$

where $\tau_R := \inf\{n \ge 0 : Y_n \notin B(0,R)\}$.

(c) $\forall \omega \in \Omega_0$ *and* $\forall x \in \mathcal{G}(\omega)$, $\exists N_x(\omega)$ *s.t.* $P_\omega^x(N_x(\omega) < \infty) = 1$ *and*

$$(\log n)^{-\alpha_3} n^{1/3} \leq \max_{0 \leq k \leq n} d(0, Y_k) \leq (\log n)^{\alpha_3} n^{1/3}, \quad \forall n \geq N_x(\omega), \ P_\omega^x - a.s.$$

(d) $c_1 R^3 \leq \mathbb{E}(E_\cdot^0 \tau_R) \leq c_2 R^3, \quad c_3 n^{-2/3} \leq \mathbb{E}(p_{2n}^\cdot(0,0)) \leq c_4 n^{-2/3}, \quad \forall R, n \geq 1.$

Remark 2.2

1. We cannot let $\alpha_1 = 0$ *in general (cf. tree case: Barlow-K '06)*

2. The critical dim. is $d = 4$. *(Open prob.) Does the RW behave similarly to the mean field case for* $d = 5, 6$*?*

3. (Related work) RW on IIC on trees

Kesten ('86): Annealed tightness and convergence of the scaled hight proc.

Barlow-K ('06): Detailed heat kernel estimates and NON quenched tightness.

Theorem 2.1 can be proved by combining analytic and probabilistic estimates.

Analytic Estimates For $\lambda > 1$, let

$$J(\lambda) := \{R \in [1,\infty] : \lambda^{-1}R^2 \le \mu(B(0,R)) \le \lambda R^2, R_{\text{eff}}(0, B(0,R)^c) \ge \lambda^{-1}R\}.$$

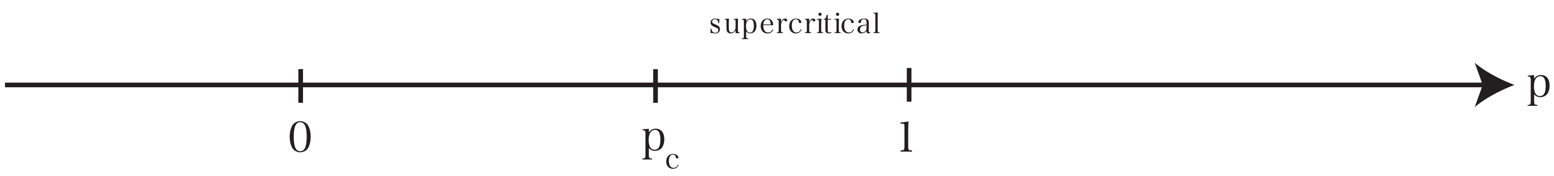
Assumption 2.3 *(1)* $\exists p(\lambda)$ *with* $p(\lambda) \le c_3\lambda^{-q_0}$ *(for* $\exists q_0, c_3 > 0$*) s.t.*

$$\mathbb{P}(R \in J(\lambda)) \ge 1 - p(\lambda), \qquad \forall R \ge 1.$$

(2) $\mathbb{E}[V(R)] \le c_1 R^2, \quad \mathbb{E}[1/V(R)] \le c_2 R^{-2}.$

If the random graph $(\mathcal{G}(\omega), \omega \in \Omega)$ satisfies Assumption 2.3, then the simple RW on the graph satisfies Theorem 2.1. (Use technique developed in analysis on fractals!)

$\Rightarrow$ prove Assumption 2.3 for the model (Probabilistic estimates)



Supercritical case ($p > p_c$)

- De Masi, Ferrari, Goldstein and Wick (1989): Inv. principle for the annealed case
- Mathieu and Remy (2004): Isoperimetric ineq. and heat kernel decay
- Barlow (2004): Detailed Gaussian heat kernel estimates

$\exists c_i = c_i(p, d) > 0$ and r.v. $S_x(\omega) < \infty$ s.t. $\forall x, y \in \mathcal{G}(\omega)$ and $S_x(\omega) \vee \|x-y\| \vee 1 \leq t$,

$$c_1 t^{-d/2} \exp\Big(-\frac{\|x-y|^2}{c_1 t}\Big) \leq p_t^\omega(x,y) \leq c_2 t^{-d/2} \exp\Big(-\frac{\|x-y\|^2}{c_2 t}\Big).$$

- Sidoravicius and A.-S. Sznitman (2004): Inv. principle for the quenched case ($d \geq 4$)

  Mathieu and Piatnitski, Berger and Biskup (Preprints, $d \geq 2$)

$$\frac{1}{\sqrt{n}} Y^\omega_{[n\,\cdot]} \stackrel{n\to\infty}{\longrightarrow} B_{\sigma\,\cdot}, \qquad \mathbb{P} - \text{a.e. } \omega.$$

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(It is now believed that this is false for small d.)

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Kesten (1986): $d = 2$ 'subdiffusive behaviour', i.e. $\exists \epsilon > 0$ s.t. $n^{-\frac{1}{2}+\epsilon} d(0, Y_n)$ is tight.

$\Rightarrow$ This motivated the study of diffusion processes on fractals.
(Related recent work) $d = 2$: Smirnov, Lawler, Schramn, Werner, ...
$\Rightarrow$ Shape of the cont. limit etc. (Very Active)