A trace theorem for Dirichlet forms on fractals and its application

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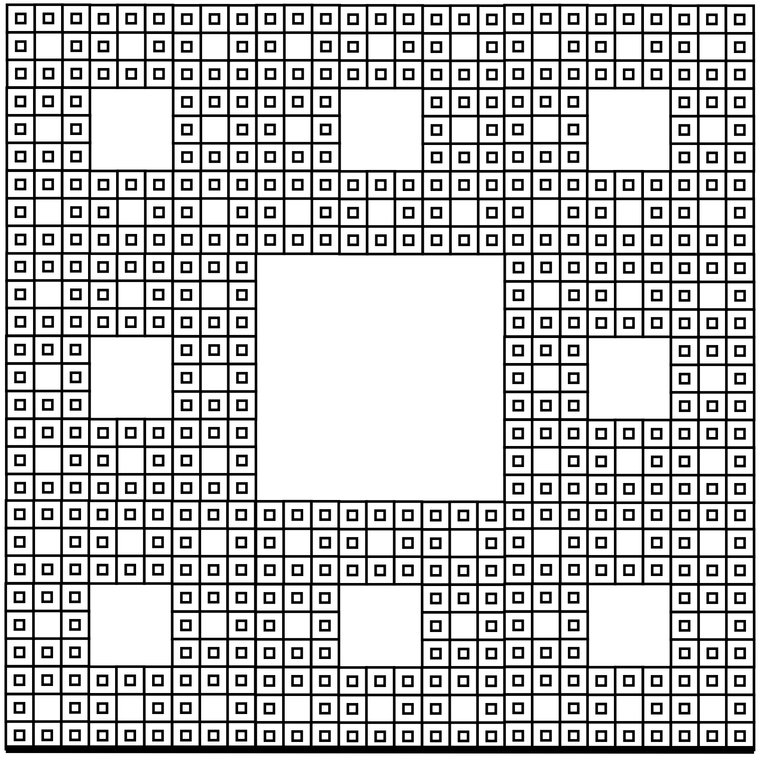
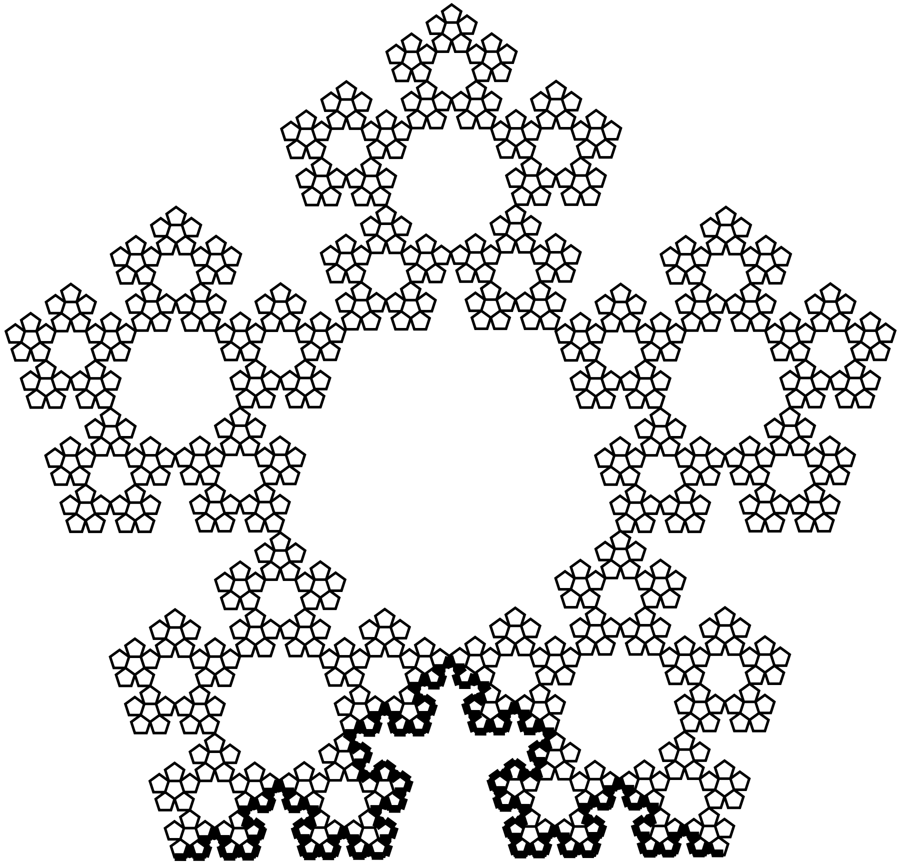
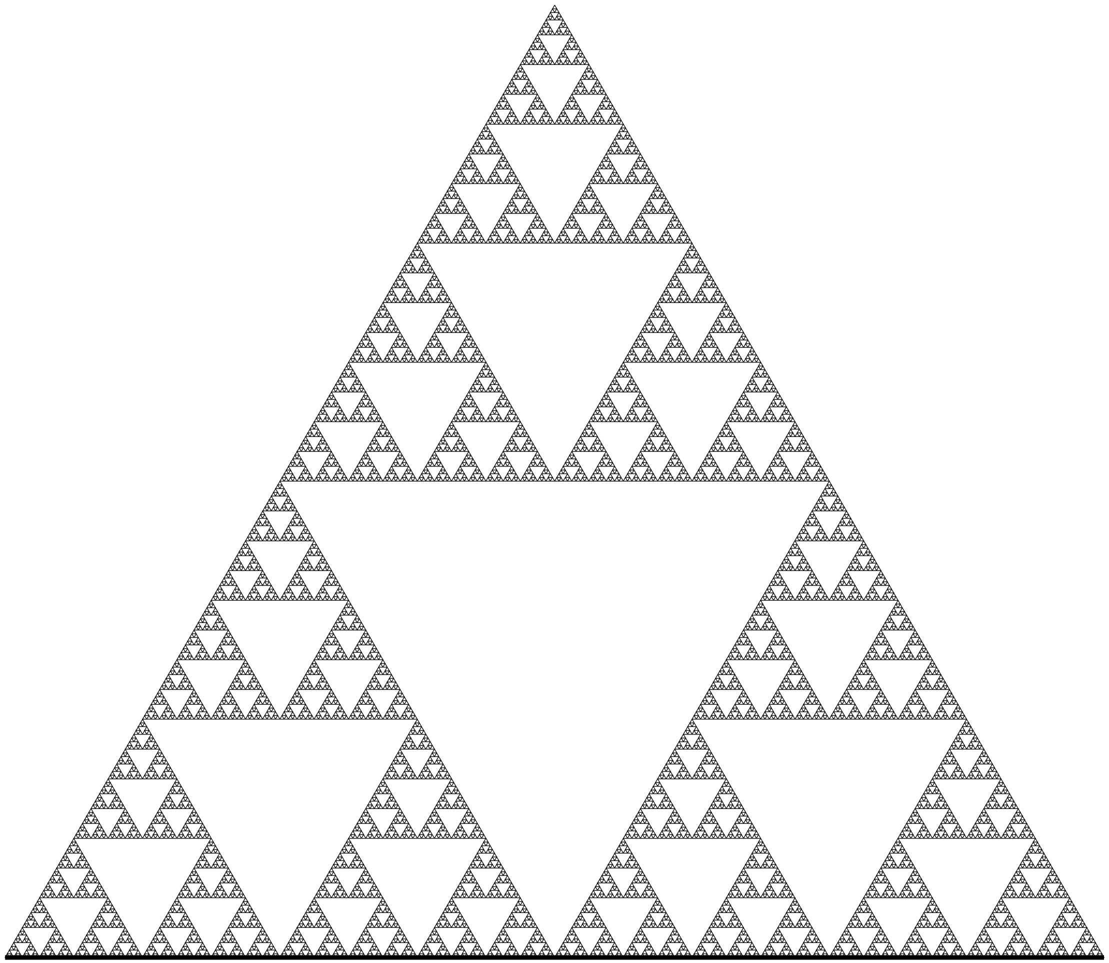




Jonsson (Math Z. '05)



1 A quick view of the theory of Dirichlet forms

General Theory (see Fukushima-Oshima-Takeda '94, Röckner-Ma '92 etc.)

$\{X_t\}_t$: Sym. Hunt proc. on (K, μ) $\oplus$ cont. path (diffusion)

$\Leftrightarrow$ $-\Delta$: non-neg. def. self-adj. op. on $\mathbb{L}^2$ s.t. $P_t := \exp(t\Delta)$ Markovian $\oplus$ local

$$P_t f(x) = E^x[f(X_t)], \ \lim_{t\to 0}(P_t - I)/t = \Delta$$

$\Leftrightarrow$ $(\mathcal{E}, \mathcal{F})$: regular Dirichlet form (i.e. sym. closed Markovian form) on $\mathbb{L}^2$

$$\mathcal{E}(u, v) = \int_K \sqrt{-\Delta}u\sqrt{-\Delta}v d\mu, \ \mathcal{F} = \mathcal{D}(\sqrt{-\Delta}) \oplus \text{local}$$

•$(\mathcal{E}, \mathcal{F})$: regular $\overset{\text{Def}}{\Leftrightarrow}$ $\exists C \subset \mathcal{F} \cap C_0(K)$ linear space which is dense

i) in $\mathcal{F}$ w.r.t. $\mathcal{E}_1$-norm and ii) in $C_0(K)$ w.r.t. $\|\cdot\|_\infty$-norm.

•$(\mathcal{E}, \mathcal{F})$: local $\overset{\text{Def}}{\Leftrightarrow}$ $(u, v \in \mathcal{F}, \mathrm{Supp}\, u \cap \mathrm{Supp}\, v = \emptyset \Rightarrow \mathcal{E}(u, v) = 0)$.

Exam. BM on $\mathbb{R}^n$ $\Leftrightarrow$ Laplace op. on $\mathbb{R}^n$ $\Leftrightarrow$ $\mathcal{E}(f, f) = \frac{1}{2}\int |\nabla f|^2 dx, \ \mathcal{F} = H^1(\mathbb{R}^n)$

2 Dirichlet forms on fractals

Sierpinski carpets $\{F_i\}_{i=1}^N : \mathbb{R}^n \to \mathbb{R}^n$; contraction maps $(N = 3^n - 1)$

$$d(F_i(x), F_i(y)) = d(x, y)/3 \qquad \forall x, y \in \mathbb{R}^n$$

$\exists 1$ non-void compact set K s.t. $K = \cup_{i=1}^N F_i(K)$.

K: (n-dimensional) Sierpinski carpet

$d_f := \log N / \log 3$: Hausdorff dimension of K (w.r.t. the Euclidean metric)

μ: (normalized) Hausdorff measure on K, i.e. a Borel measure on K s.t.

$$\mu(F_{i_1 \cdots i_n}(K)) = 3^{-n} \qquad \forall i_1, \cdots, i_n \in S := \{1, \cdots, N\}.$$

Theorem 2.1 (Barlow-Bass, Kusuoka-Zhou, etc.)

$\exists(\mathcal{E}, \mathcal{F})$: *a local regular Dirichlet form on* $\mathbb{L}^2(K, \mu)$ *s.t.* $\exists \rho > 0$

$$\mathcal{E}(f, g) = \rho \sum_{i \in S} \mathcal{E}(f \circ F_i, g \circ F_i), \quad \forall f, g \in \mathcal{F} \quad \text{(Self-similarity)}.$$

(Property I) Heat kernel estimates

Theorem 2.2 (Barlow-Bass) *Let* $d_w = \log(\rho N)/\log 3 > 2$.

$\exists p_t(x,y)$: *jointly continuous sym. transition density of* $\{X_t\}$ *w.r.t.* μ

$(P_t f(x) = \int_K p_t(x,y) f(y)\mu(dy)\ \forall x \in K,\quad \frac{\partial}{\partial t} p_t(x_0,x) = \Delta_x p_t(x_0,x)\ )$ *s.t.*

$$c_1 t^{-d_f/d_w} \exp(-c_2(\frac{d(x,y)^{d_w}}{t})^{\frac{1}{d_w-1}}) \le p_t(x,y) \le c_3 t^{-d_f/d_w} \exp(-c_4(\frac{d(x,y)^{d_w}}{t})^{\frac{1}{d_w-1}}). \quad (2.1)$$

By integrating (2.1), we have $E^0[d(0,X_t)] \asymp t^{1/d_w}$.

As $d_w > 2$, we say the process is sub-diffusive.

Remark. $\rho > 1$ for $n = 2$, $\rho < 1$ for $n \ge 3$.

(Property II) Domains of the Dirichlet forms

For $1 \leq p < \infty,\ 1 \leq q \leq \infty,\ \beta \geq 0$ and $m \in \mathbb{N} \cup \{0\}$, set

$$a_m(\beta, f) := L^{m\beta}(L^{md_f} \int\int_{|x-y|<c_0L^{-m}} |f(x) - f(y)|^p d\mu(x)d\mu(y))^{1/p}, \quad f \in \mathbb{L}^p(K, \mu),$$

where $1 < L < \infty,\ 0 < c_0 < \infty$.

$\Lambda^{\beta}_{p,q}(K)$: a set of $f \in \mathbb{L}^p(K, \mu)$ s.t. $\bar{a}(\beta, f) := \{a_m(\beta, f)\}_{m=0}^{\infty} \in l^q$.

$\Lambda^{\beta}_{p,q}(K)$ is a *Besov-Lipschitz space*. It is a Banach space.

$\underline{p = 2}$ $\Lambda^{\beta}_{2,q}(\mathbb{R}^n) = B^{\beta}_{2,q}(\mathbb{R}^n)$ if $0 < \beta < 1$, $= \{0\}$ if $\beta > 1$.

$\underline{p = 2, \beta = 1}$ $\Lambda^{1}_{2,\infty}(\mathbb{R}^n) = H^1(\mathbb{R}^n)$, $\Lambda^{1}_{2,2}(\mathbb{R}^n) = \{0\}$.

Theorem 2.3 (Jonsson '96 (for S. gasket), K, Paluba, Grigor'yan-Hu-Lau, K-Sturm)

Let $(\mathcal{E}, \mathcal{F})$ be the Dirichlet form on the gasket. Then,

$$\mathcal{F} = \Lambda^{d_w/2}_{2,\infty}(K).$$

(Q) $\mathrm{Tr}_L\, \mathcal{F} = ?, \quad L = [0, 1]^{n-1} \times \{0\}.$

Recall that for $\alpha > 0$ and $k \in \mathbb{N}$ where $k < \alpha \leq k+1$,

$$B^{\alpha}_{p,q}(\mathbb{R}^n) := \{u \in L^p(\mathbb{R}^n, m) : \|u\|_{B^{\alpha}_{p,q}} < \infty\},$$

$$\text{where} \quad \|u\|_{B^{\alpha}_{p,q}} := \sum_{0 \leq |j| \leq k} \|D^j u\|_{L^p} + \sum_{|j|=k} \Big(\int_{\mathbb{R}^n} \frac{\|\Delta_h D^j u\|^q_{L^p}}{|h|^{n+q(\alpha-k)}} dh \Big)^{1/q}. \quad --(*)$$

Here, $j = (j_1, \ldots, j_n)$, $|j| = j_1 + \cdots + j_n$, $D^j = \frac{\partial^{|j|}}{\partial x_1^{j_1} \cdots \partial x_n^{j_n}}$,

and $\Delta_h f(x) := f(x+h) - f(x)$.

When $\alpha \in \mathbb{N}$, Δ_h in (*) is changed to Δ_h^2.

3 A trace theorem for Dirichlet forms on fractals

Theorem 3.1 (Hino-K '05) *Let d be the Hdff dim. for L (in this case $n-1$). Then,*

$$Tr_L \mathcal{F} = \Lambda^{\beta}_{2,2}(L), \quad \beta = \frac{d_w}{2} - \frac{d_f - d}{2}.$$

<u>What is Tr_L?</u>

$\forall f \in \mathcal{F}$, $\exists \tilde{f}$: quasi-cont. modification of f

(I.e. $f = \tilde{f}$, μ-a.e., $\forall \epsilon > 0$, $\exists G \subset K, \mathrm{Cap} G < \epsilon$ s.t. $\tilde{f}|_{K \setminus G}$ is finite and cont.)

Let ν be the Hdff meas. on L (in this case $(n-1)$-dim. Lebesgue meas.).

Since $d_w > d_f - d$, ν charges no set of 0-capacity (i.e. $\mathrm{Cap} A = 0 \Rightarrow \nu(A) = 0$).

So, $\tilde{f}$ is determined ν-a.e. on L. Further, $\tilde{f}|_L \in \mathbb{L}^2(L, \nu)$.

Thus, $\mathrm{Tr}_L \mathcal{F} := \{\tilde{f}|_L : f \in \mathcal{F}\}$.

More generally, (X, d, μ) 'nice' metric meas. space

$\{F_i\}_{i\in S} : X \to X, \quad \exists\alpha > 1$ s.t. $d(F_i(x), F_i(y)) = \alpha^{-1} d(x, y), \ \forall x, y \in X$

$\exists 1$ non-void compact set K s.t. $K = \cup_{i\in S} F_i(K)$, μ: Hdff meas., d_f: Hdff dim.

Assume $\exists(\mathcal{E}, \mathcal{F})$: loc. reg. D-form on $\mathbb{L}^2(K, \mu)$ that satisfies

• Self-similarity • Elliptic Harnack ineq. • Poincaré ineq.

L: self-similar subset $\{F_i\}_{i\in I}$, $I \subset S$, $L = \cup_{i\in I} F_i(L)$, ν:Hdff meas., d: Hdff dim. s.t.

(1) $d_w > d_f - d$ & $\nu(D) \le c\mathrm{Cap}(D)$ $\forall D$: compact in K.

(2) "Technical" geometric assumptions on K and L

Assumption A $\quad f \in \mathcal{F}$, $\mathcal{E}_{S^m \setminus I^m}(f, f) = 0\ \forall m \Rightarrow f \equiv$ const.

Here, $\mathcal{E}_A(f, f) := \rho^m \sum_{w\in A} \mathcal{E}(f \circ F_w, f \circ F_w)$ for $A \subset S^m$.

Theorem 3.2 (Hino-K '05) *Under above conditions,*

$$Tr_L\, \mathcal{F} = \overline{\Lambda^{\beta}_{2,2}(L) \cap C_0(L)}^{\|\cdot\|_{\Lambda^{\beta}_{2,2}(L)}}, \qquad \beta = \frac{d_w}{2} - \frac{d_f - d}{2}. \tag{3.1}$$

Remarks.

- The latter half of (1) holds if $K \subset \mathbb{R}^n$ & $\mathcal{F} = \Lambda^{d_w/2}_{2,\infty}(K)$

  or the diffusion for $(\mathcal{E}, \mathcal{F})$ satisfies the heat kernel estimates (2.1).

- (RHS of (3.1))$= \Lambda^{\beta}_{2,2}(L)$ if (a) $L \subset \mathbb{R}^D, \beta < 1$ or (b) $\beta > d/2$.

By the general theory of D-forms, we have the reg. D-form $(\check{\mathcal{E}}, \check{\mathcal{F}})$ on $\mathbb{L}^2(L, \nu)$, the trace of $(\mathcal{E}, \mathcal{F})$ to L by ν. The corresponding process is in general a jump-type process.

(Classical case: $\mathrm{Tr}_{\mathbb{R}^n} H^1(\mathbb{R}^n) = B^{2,2}_{1/2}(\mathbb{R}^{n-1})$ domain of the D-form for the Cauchy proc.)

The above theorems characterize the domain $\check{\mathcal{F}} = \mathrm{Tr}_L \mathcal{F}$.

Examples.

- K: n-dim. Sierpinski gaskets, L: $(n-1)$-dim. gasket on the bottom

  (Jonsson (Math Z. '05), $n=2$ case)

$$\beta = \frac{\log(n+3)}{2\log 2} - \frac{\log(1+1/n)}{2\log 2}.$$

- Nested fractals

  E.g., K: Penta-kun, L either the Cantor set or the Koch-like curve in K.

- K Vicsek set, L: diagonal line

  Assumption A does not hold!

$$\mathrm{Tr}_L\,\mathcal{F} = \Lambda^1_{2,\infty}(L) : \text{ Brownian motion on } L$$

In general, $\Lambda^\beta_{2,2}(L) \subset \mathrm{Tr}_L\,\mathcal{F} \subset \Lambda^\beta_{2,\infty}(L)$, but we cannot say anymore.

Trace from $\mathbb{R}^n$ to a d-set, $\quad \beta > 0$

Triebel ('97 book) No extension thm

- $\mathbb{B}^{\beta}_{p,q}(K) := tr_K B^{\beta+(n-d)/2}_{p,q}(\mathbb{R}^n) \subset \mathbb{L}^p(K,\mu)$
- $\|f|\mathbb{B}^{\beta}_{p,q}(K)\| := \inf_{tr_K\, g=f} \|g|B^{\beta+(n-d)/2}_{p,q}(\mathbb{R}^n)\|,$

Jonsson-Wallin ('84 book) Both restriction and extension thm

- $B^{p,q}_{\beta,JW}(K) := \{\{f^{(j)}\}_{|j|\leq k} : \cdots\} \leftarrow$ Def. quite involved, Vector space

  $f^{(j)}$: formal j-th derivative of f, j: multi index, $\quad k < \beta \leq k+1,\ k \in \mathbb{N}$

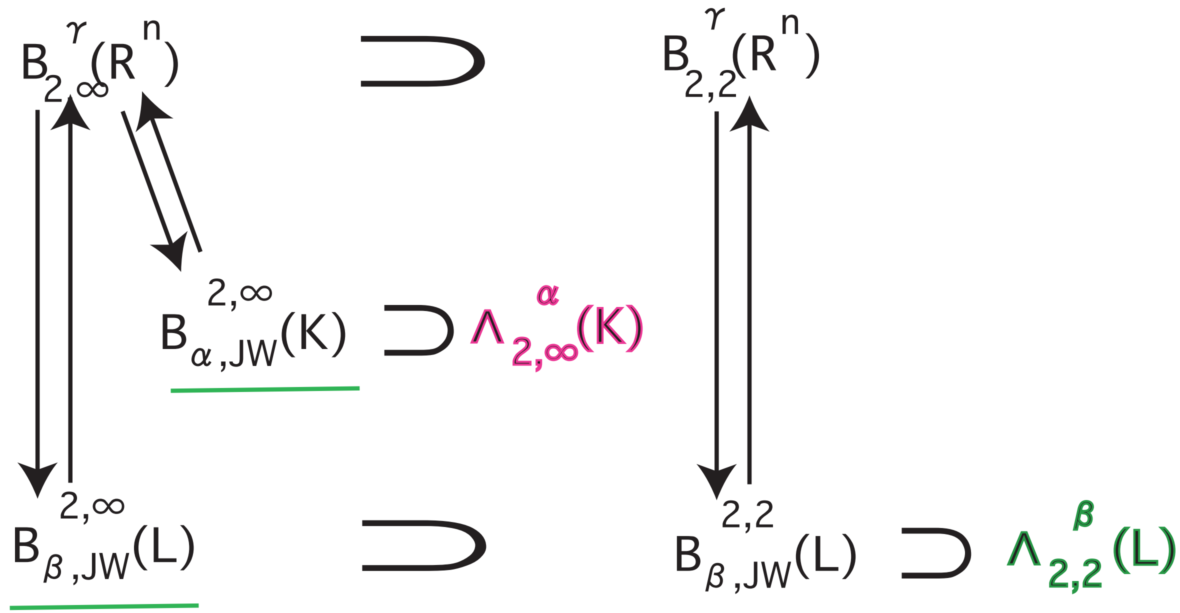
- $B^{p,q}_{\beta,JW}(K) = B^{\beta++(n-d)/2}_{p,q}(\mathbb{R}^n)|_K$

  $\Lambda^{\beta}_{p,q}(K) \subset B^{p,q}_{\beta,JW}(K)$ in the sense $f \mapsto (f,0,0,\ldots,0)$

When $\beta < 1$, both def. coincides and $\mathbb{B}^{\beta}_{p,q}(K) = B^{p,q}_{\beta,JW}(K) = \Lambda^{\beta}_{p,q}(K)$.

Our thm cannot be obtained from these.

Consider the case $K \subset R^n$



4 Application: Penetrating processes

K_i: fractal $(i = 1, 2)$, $G = K_1 \cup K_2$

Assume $\exists(\mathcal{E}_{K_i}, \mathcal{F}_{K_i})$ on $\mathbb{L}^2(K_i, \mu_i)$: loc. reg. D-form on K_i with 'nice' properties.

(Q) Construct diffusion on G which behave as the appropriate diff. on each K_i.

Superposition of D-forms $\qquad \tilde{\mu} = \mu_1 + \mu_2$

$$\tilde{\mathcal{E}}(u, v) := \mathcal{E}_{K_1}(u|_{K_1}, v|_{K_1}) + \mathcal{E}_{K_2}(u|_{K_2}, v|_{K_2}),$$

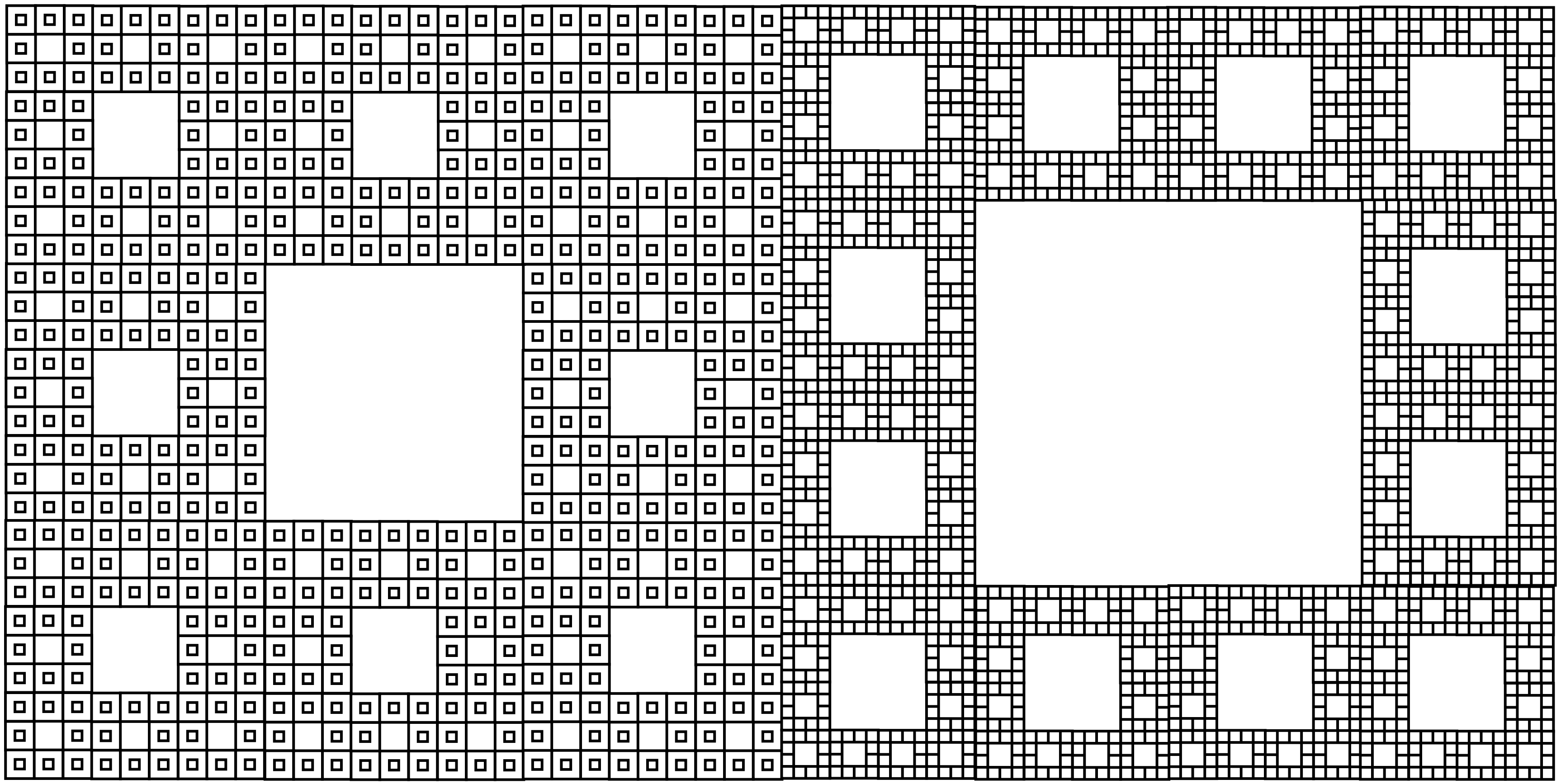
$$\mathcal{D}(\tilde{\mathcal{E}}) := \{u \in C_0(G) : u|_{K_i} \in \mathcal{F}_{K_i},\ i = 1, 2,\ \ \tilde{\mathcal{E}}(u, u) < \infty\}.$$

(Q) Enough functions in $\mathcal{D}(\tilde{\mathcal{E}})$? Esp., $\mathcal{D}(\tilde{\mathcal{E}})$ dense in $C_0(G)$? $\Leftarrow$ Using trace thm, YES!

Once solved, $(\tilde{\mathcal{E}}, \tilde{\mathcal{F}})$ loc. reg. D-form on $\mathbb{L}^2(G, \tilde{\mu})$ where $\tilde{\mathcal{F}} = \overline{\mathcal{D}(\tilde{\mathcal{E}})}^{\tilde{\mathcal{E}}_1}$.

Various properties of the diffusion can be obtained. (K, JFA '00; Hambly-K, PTRF '03)

Penetrating $\tilde{P}^x(\sigma_B < \infty) > 0$ q.e. $x \in G$, $\forall B$: pos. cap.



Theorem 4.1 (Hambly-K '03) *Short time heat kernel estimates*

Under the following strong assumption

$$\frac{2}{d_s(K_i)} - \frac{2}{d_f(K_i)d_c(K_i)} < \kappa \qquad i = 1, 2, \tag{4.1}$$

we have the following for all $x, y \in G$.

$$p_t(x,y) \le c_1 t^{-(d_s(x)\vee d_s(y))/2}\Phi(d^{(1)}(x,y), d^{(2)}(x,y), c_2 t), \qquad \forall t < \exists t_0(x) \wedge t_0(y),$$
$$p_t(x,y) \ge c_3 t^{\theta(x,y)}\Xi(x,y,t)\Phi(d^{(1)}(x,y), d^{(2)}(x,y), c_4 t), \qquad \forall t < 1,$$

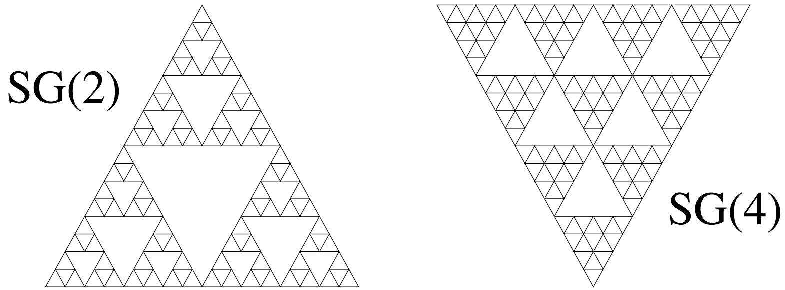
where

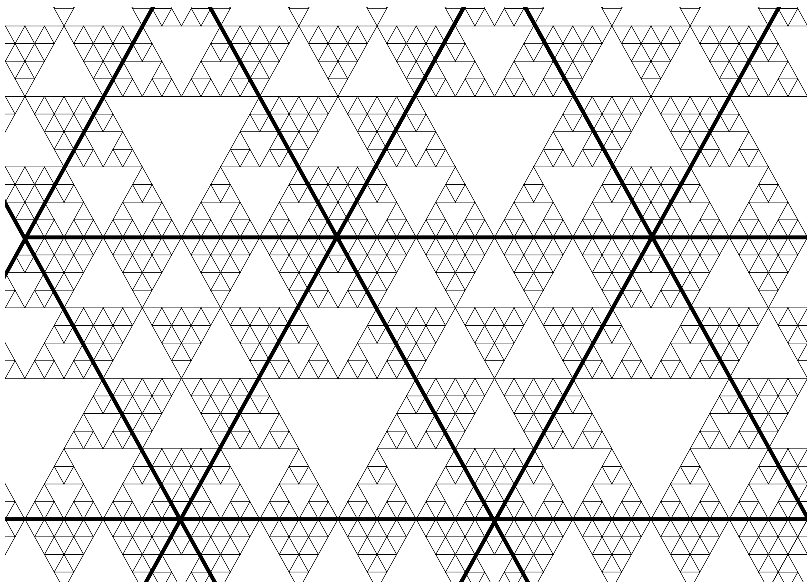
$$\Phi(u_1, u_2, t) = \exp(-\sum_{i=1}^{2}(\frac{u_i^{d_w^{(i)}}}{t})^{1/(d_w^{(i)}-1)}).$$

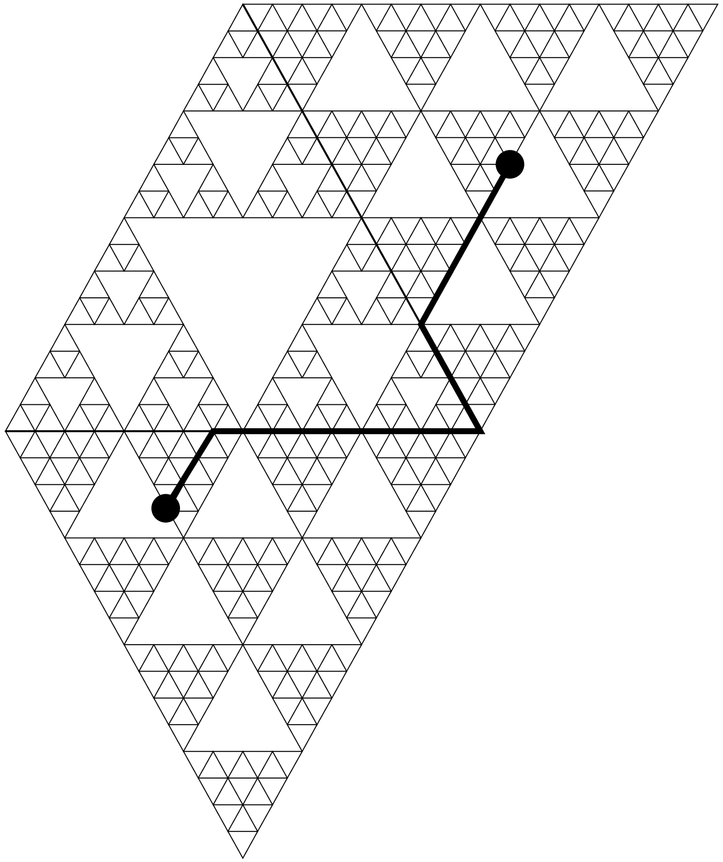
θ, Ξ: error terms. When $x = y$, $\Xi(x,y,t) = 1$ and $\theta(x,y) = -(d_s(x) \vee d_s(y))/2$.

Only nested fractals, needed strong assumption (4.1). $\Rightarrow$ The trace thm. generalizes this.

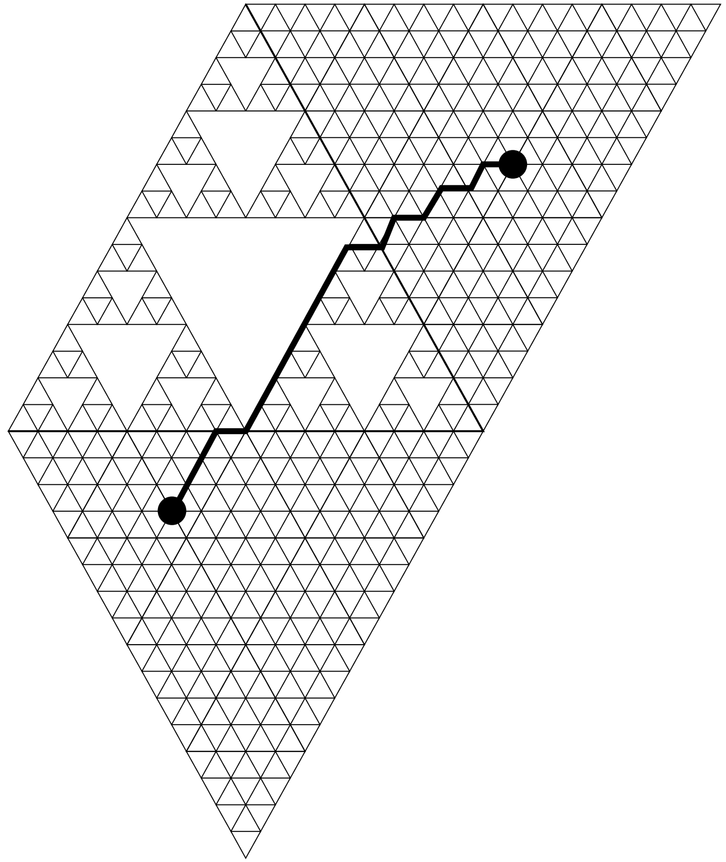
Fractal tiling (Hambly-K '03)







$d_w^{(2)} < d_w^{(4)}$



$d_w^{(2)} > d_w^{(l)}$: l large

"Most probable path avoids to move on K_i where $d_w^{(i)}$ is small."

5 Ideas of the proof of the trace thm

Basic idea: Use self-similarity instead of the differential structure!

(Discrete Approximation)

- $Q_n : \mathbb{L}^1(L,\nu) \to \mathbb{R}^{I^n}, \qquad Q_n f(w) = ⨍_{F_w(L)} f(s)d\nu(s).$
- $E_{(n)}(g) := \sum_{v,w\in I^n, v \overset{n}{\sim} w} (g(v)-g(w))^2, \quad \forall g \in \mathbb{R}^{I^n}.$

Lemma 5.1 *1)* $\|\bar{a}(\beta,f)\|_{l^2}^2 \asymp \sum_{n=1}^{\infty} \alpha^{(2\beta-d)n} E_{(n)}(Q_n f) = \sum_{n=1}^{\infty} \rho^n E_{(n)}(Q_n f)$

2) $E_{(n)}(Q_n f) \le c\rho^{-n} \mathcal{E}_{I^n}(f,f), \quad \forall f \in \mathcal{F}, \ \forall n.$

Let $\mathcal{H}_{I^n} := \{h \in \mathcal{F} : \mathcal{E}(h,f) = 0, \ \ \forall f \in \mathcal{F}_{I^n}\}$

where $\mathcal{F}_{I^n} = \{f \in \mathcal{F} : f = 0 \text{ on } K_{S^n\setminus I^n}, Q_n f \equiv 0\}.$

Lemma 5.2 (Key Lemma) *Under Assumption A,* $\exists c_0 < 1$ *s.t.*

$$\mathcal{E}_{I^{n+1}}(h,h) \le c_0 \mathcal{E}_{I^n}(h,h), \qquad \forall h \in \mathcal{H}_{I^n}.$$

(Restriction thm) For each $f \in \mathcal{F}$, let $g_n \in \mathcal{F}$ be s.t.

$g_n = f$ on $K_{S^n \setminus I^n}$, $Q_n g_n = Q_n f$, $g_n \in \mathcal{H}_{I^n}$. Then $g_n \to f$ in $\mathcal{F}$.

Let $f_n := g_n - g_{n-1}$ ($g_{-1} \equiv 0$). Then $f = \sum_n f_n$,

$\mathcal{E}(f_i, f_j) = 0$, $i \neq j$ (ortho. decomp.). So $\mathcal{E}(f, f) = \sum_{n=0}^{\infty} \mathcal{E}(f_n, f_n)$. Using these,

$$(E_{(n)}(Q_n f))^{1/2} = (E_{(n)}(Q_n g_n))^{1/2} = (E_{(n)}(\sum_{j=0}^{n} Q_n f_j))^{1/2}$$

$$\overset{\text{Minkowski}}{\leq} \sum_{j=0}^{n} (E_{(n)}(Q_n f_j))^{1/2} \overset{\text{Lem 5.1 2)}}{\leq} c \sum_{j=0}^{n} (\rho^{-n} \mathcal{E}_{I^n}(f_j))^{1/2} \overset{\text{Lem 5.2}}{\leq} c' \sum_{j=0}^{n} (\rho^{-n} c_0^{n-j} \mathcal{E}_{I^j}(f_j))^{1/2}.$$

So we obtain
$$\sum_{n=0}^{\infty} \rho^n E_{(n)}(Q_n f) \leq c \sum_{n=0}^{\infty} c_0^n (\sum_{j=0}^{n} (c_0^{-j} \mathcal{E}_{I^j}(f_j))^{1/2})^2$$
$$\overset{(*)}{\leq} c' \sum_{j=0}^{\infty} c_0^j c_0^{-j} \mathcal{E}(f_j) = c' \sum_j \mathcal{E}(f_j) = c' \mathcal{E}(f),$$

where $\sum_{i=0}^{\infty} 2^{ai} (\sum_{j=0}^{i} a_j)^p \leq c \sum_{j=0}^{\infty} 2^{aj} a_j^p$ for $a < 0, p > 0, a_j \geq 0$ is used in (*) □

Remark. The fact that $F_v^* : \mathcal{H}(F_v(K))|_{F_v(K)} \to \mathcal{F}$ is a compact operator is used (only) in Lem 5.2. (Here $F_v^* h = h \circ F_v$.)

Elliptic Harnack ineq. is used to guarantee this.

(Extension thm) Similar to the classical ext. thm (Whitney decomp., assoc. partition of unity etc.), using self-similarity.