# Lecture 3

# Resistance forms and Heat kernels

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## § 1 **Resistance forms on MMD and weighted graphs**

$(X, d, \mu)$ metric measure space (MM space)

$(X, d)$: connected loc. cpt complete sep. metric space

$d$: geodesic

$\mu$: Borel meas. on $X$ s.t. $0 < \mu(B) < \infty$, $\forall$ball $B$.

Denote $B(x, r) = \{y : d(x, y) < r\}$, $V(x, r) = \mu(B(x, r))$.

For simplicity, assume $X$ has infinite diameter.

$(X, d, \mu, \mathcal{E})$: metric measure Dirichlet space (MMD space)

$(X, d, \mu)$: MM space

$(\mathcal{E}, \mathcal{F})$: regular, strong local Dirichlet form on $L^2(X, \mu)$

$\Rightarrow$ $-\Delta$: self-adjoint operator, $\{P_t\}$: semigroup,

$Y = \{Y_t, t \geq 0, P^x, x \in X\}$: diffusion

Assume $(\mathcal{E}, \mathcal{F})$ is conservative (stochastically complete)

(i.e. $P_t 1 = 1, \forall t > 0$).

$(X,\mu)$: weighted infinite graph

$\{\mu_{xy}\}_{x,y\in X}$: conductance, $\mu_{xy}=\mu_{yx}\geq 0$, $\mu_{xy}>0 \Leftrightarrow x\sim y$

$$\mathcal{E}(f,f):=\frac{1}{2}\sum_{x\sim y}(f(x)-f(y))^2\mu_{xy}, \qquad \forall f\in L^2(X,\mu).$$

$$\mu_x:=\sum_y \mu_{xy}, \quad \mu(A):=\sum_{x\in A}\mu_x$$

$Y=(Y_n,P^x,x\in X)$: Markov chain,

$P(x,y):=\mu_{xy}/\mu_x$, $\quad d(\cdot,\cdot)$: graph distance on $X$

We assume the following.

$p_0$-condition: $\exists p_0>0$, $\quad P(x,y)=\mu_{xy}/\mu_x\geq p_0 \quad \forall x\sim y$.

$(VD)$: volume doubling condition

$V(x,2R)\leq c_1V(x,R) \qquad \forall\ x\in X, R\geq 1$

## Resistance forms on $(X, d, \mu, \mathcal{E})$

$(X, d, \mu, \mathcal{E})$: MMD sapce or weighted graph

It is called a resistance form if $\mathcal{F} \subset C(X)$ and

$$\sup \{ \frac{|u(p) - u(q)|^2}{\mathcal{E}(u, u)} : u \in \mathcal{F}, \mathcal{E}(u, u) > 0 \} < \infty, \qquad (1)$$

forall $p \neq q \in X$.

Define $R(p, q) =$ (LHS of (1)) if $p \neq q$ and $R(p, p) = 0$.

$R$ is called a resistance metric. By (1),

$(*)$ $|f(x) - f(y)|^2 \leq R(x, y)\mathcal{E}(f, f), \quad \forall f \in \mathcal{F}$ **Key!**

$$R(p, q) = \left( \inf\{ \mathcal{E}(f, f) : f(p) = 1, f(q) = 0, f \in \mathcal{F} \} \right)^{-1}.$$

Indeed, by linear transform $f(x) = \alpha u(x) + \beta$, we can take $f(x) = 1, f(y) = 0$ if $u$ is not const. So,

$$\begin{aligned} R(x, y) &= \sup \{ \frac{|u(x) - u(y)|^2}{\mathcal{E}(u, u)} : u \in \mathcal{F}, \mathcal{E}(u, u) > 0 \} \\ &= \sup \{ \frac{1}{\mathcal{E}(f, f)} : f \in \mathcal{F}, f(x) = 1, f(y) = 0 \} \\ &= \left( \inf\{ \mathcal{E}(f, f) : f(x) = 1, f(y) = 0, f \in \mathcal{F} \} \right)^{-1} \end{aligned}$$

**Examples**

- Dirichlet forms on the Sierpinski gasket
- Dirichlet forms on nested fractals

  (more generally, on p.c.f. self-similar sets)
- Dirichlet forms on the 2-dim. Sierpinski carpet

Dirichlet forms on 2-dim. S.G.

$F_1(x) = \frac{1}{2}x,\ F_2(x) = \frac{1}{2}(x+a_1),\ F_3(x) = \frac{1}{2}(x+a_2),\ x \in \mathbb{R}^2$

$a_1 = (1,0), a_2 = (\frac{1}{2}, \frac{\sqrt{3}}{2})$

$G$: 2-dim (compact) S.G. $G = \cup_{i=1}^3 F_i(G)$

$V_n$ vertices of triangles in $G$ with length $2^{-n}$

$$\begin{aligned}&\tilde{\mathcal{E}}_n(f,f)\\ &= (\frac{5}{3})^n \sum_{i_1,\cdots,i_n=1}^{3} \sum_{x,y\in V_0} (f \circ F_{i_1\cdots i_n}(x) - f \circ F_{i_1\cdots i_n}(y))^2\end{aligned}$$

$\tilde{\mathcal{E}}(f,f) := \lim_{n\to\infty} \tilde{\mathcal{E}}_n(f,f), \tilde{\mathcal{F}} = \{f : \tilde{\mathcal{E}}(f,f) < \infty\}$

$f \in \tilde{\mathcal{F}}$ can be extended cont. to $K$.

$(\tilde{\mathcal{E}}, \tilde{\mathcal{F}})$ local regular D-form on $L^2(G, \tilde{\mu})$.

$X = \cup_{l\geq 0} 2^n G$: unbdd 2-dim. S.G. $\sigma_l f := f \circ F_1^{-l}$

$\mathcal{E}(f,f) := \lim_{l\to\infty} (\frac{3}{5})^l \tilde{\mathcal{E}}(\sigma_l f, \sigma_l f), \mathcal{F} = \{f : \mathcal{E}(f,f) < \infty\} \cap L^2$

$(\mathcal{E}, \mathcal{F})$ local regular D-form on $L^2(X, \mu)$.

$R(\cdot,\cdot)$ well-defined and $R(F_1^l(x), F_1^l(y)) = (\frac{3}{5})^l R(x,y)$.

## § 2 **Inequalities**

- Poincaré inequality: $(PI(\beta)), \beta > 0$

$\forall B = B(x, R) \subset X$ and $\forall f \in \mathcal{F}$,

$$\int_B (f(x) - \bar{f}_B)^2 d\mu(x) \leq c_2 R^\beta \int_B d\Gamma(f, f) \qquad (PI(\beta))$$

where $\bar{f}_B = \mu(B)^{-1} \int_B f(x) d\mu(x)$.

- (Sub-)Gaussian heat kernel estimates: $(HK(\beta))$

$\exists$ Jointly continuous transition density $p_t(x, y)$ i.e.

$$P_t f(x) = \int_K p_t(x, y) f(y) \mu(dy) \ \ \forall x \in X, f \in \mathbb{L}^2$$

s.t. $\forall x, y \in X, t > 0$,

$$\frac{c_1}{\mu(x, t^{1/\beta})} \exp(-c_2 (\frac{d(x, y)^\beta}{t})^{\frac{1}{\beta - 1}}) \leq p_t(x, y)$$

$$\frac{c_3}{\mu(x, t^{1/\beta})} \exp(-c_4 (\frac{d(x, y)^\beta}{t})^{\frac{1}{\beta - 1}}). \quad (HK(\beta))$$

Note: If $(HK(\beta))$ holds, then $\beta \geq 2$.

• (Generalized) parabolic Harnack inequality: $(PHI(\beta))$

Let
$$\begin{aligned} Q_- &= (s+r^\beta, s+2r^\beta) \times B(x_0,r), \\ Q_+ &= (s+3r^\beta, s+4r^\beta) \times B(x_0,r), \\ Q &= (s, s+4r^\beta) \times B(x_0,2r). \end{aligned}$$

$\exists c_1 > 0$ s.t., $\forall u : Q \to \mathbb{R}_+$ with $\frac{\partial u}{\partial t} = \Delta u$,

$$\sup_{(t,x)\in Q_-} u(t,x) \le c_1 \inf_{(t,x)\in Q_+} u(t,x). \qquad (PHI(\beta))$$

## § 3 **Main theorem**

<u>$(VG(\beta_-))$</u>: $\exists \alpha < \beta$ s.t. $\forall x \in X, \forall r \geq s > 0$,

$$V(x,r) \leq C\left(\frac{r}{s}\right)^\alpha V(x,s).$$

<u>$(R(\beta))$</u>: $\forall x, y \in X$,

$$c_1 \frac{d(x,y)^\beta}{\mu(B(x,d(x,y)))} \leq R(x,y) \leq c_2 \frac{d(x,y)^\beta}{\mu(B(x,d(x,y)))}. \quad (2)$$

$(RL(\beta))$ if the first ineq. of (2) holds.

**Theorem 1** (Barlow-Coulhon-K '04)

*$(X, d, \mu, \mathcal{E})$: Resistance form on MMD sapce or weighted graph. Assume $(VG(\beta_-))$. Then,*

$$(HK(\beta)) \Leftrightarrow (R(\beta)) \Leftrightarrow (RL(\beta)) + (PI(\beta)). \quad (3)$$

When (3) holds, it is strongly recurrent: $\exists p_1 > 0$ s.t.

$$P^x(T_y < \tau_{B(x,2r)}) \geq p_1, \ \forall x \in X, r > 0, \ y \in B(x,r),$$

$$T_A = \inf\{t \geq 0 : X_t \in A\}, \quad \tau_A = \inf\{t \geq 0 : X_t \notin A\}.$$

- The D-form on 2-dim S.G. satisfies (3) with $\beta = \log 5/\log 2$. ($N$-dim S.G. $\log(N+3)/\log 2$).
- Dirichlet forms on nested fractals satisfy (3).
- Dirichlet forms on the 2-dim. S.C. satisfy (3).

**Corollary 2** $(X, \mu)$: *weighted graph with* $c_1 \le \mu_{xy} \le c_2$ *for all* $x \sim y$. *Assume that* $X$ *is a tree. Then,*

$$(VG(\beta_-))+(HK(\beta)) \Leftrightarrow [V(x,d(x,y)) \asymp d(x,y)^{\beta-1} \ \forall x,y].$$

Idea of the proof of $(R(\beta)) \Rightarrow (HK(\beta))$

Step 1: Nash-type ineq. $p_t(x,y) \le \frac{c_1}{\mu(x,t^{1/\beta})}$

(To be explained on the board.)

Step 2: $c_2 r^\beta \le E^y[\tau_{B(x,r)}] \le c_3 r^\beta, \ \forall y \in B(x,r/2) \quad (E_\beta)$

$\Leftarrow \frac{c_4 r^\beta}{V(x,r)} \le g_B(x,x) \le \frac{c_5 r^\beta}{V(x,r)}$ (To be expl. on the board.)

$+(*)$

Step 3: $P^x(\tau_{B(x,r)} \le t) \le c_6(\frac{r^\beta}{t})^{1/(\beta-1)}$ (Next page.)

$\Rightarrow$ From Step1 $\sim$ Step 3, upper bound can be obtained.

Step 4: $p_t(x,y) \ge \frac{c_7}{\mu(x,t^{1/\beta})}$ for $d(x,y) \le c_8 t^{1/\beta}$.

$\Leftarrow$ Upper bound $+(*)$ etc.

Step 5: Lower bound $\Leftarrow$ Step 4 + chain argument.

**Lemma 3** *(Barlow-Bass)* $\{\xi_i\}$ *non-neg. r.v.s s.t.*

$$P(\xi_i \leq t | \sigma(\xi_1, \cdots, \xi_{i-1})) \leq p + at, \quad 0 < \exists p < 1, \exists a > 0$$

$$\Rightarrow \log P(\textstyle\sum_{i=1}^n \xi_i \leq t) \leq 2(\frac{ant}{p})^{1/2} - n \log \frac{1}{p}.$$

From $(E_\beta)$ and Markov prop.,

$$c_1 r^\beta \leq E^x \tau_B \leq t + E^x[1_{\{\tau_B > t\}} E^{X_t} \tau_B] \leq t + c_2 r^\beta P^x(\tau_B > t).$$

So, $P^x(\tau_B \leq t) \leq p + at/r^\beta, \quad 0 < \exists p < 1, \exists a > 0.$

Fix $l \in \mathbb{N}$, let $b = [r/l]$,

$\sigma_{i+1} := \inf\{t \geq \sigma_i : d(X_{\sigma_i}, X_t) \geq b\}$ $(\sigma_0 = 0)$,

$\xi_i := \sigma_i - \sigma_{i-1}$. Then $\sigma_l = \sum_{i=1}^l \xi_i \leq \tau_{B(x,r)}$.

By Lemma 3, $\log P^x(\tau_B \leq t) \leq c_1(\frac{l^{\beta+1} t}{r^\beta})^{1/2} - c_2 l$.

Optimize $l$ by $l^{\beta-1} \asymp r^\beta/t$.

$\Rightarrow P^x(\tau_{B(x,r)} \leq t) \leq c_6(\frac{r^\beta}{t})^{1/(\beta-1)}$.