Convergence of symmetric Markov chains on $\mathbb{Z}^d$

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1 Introduction

Let $V_k := k^{-1}\mathbb{Z}^d$.

(Q1) Given a 'nice' Hunt process X on $\mathbb{R}^d$, can one approximate X

by a suitable family of Markov chains $X^{(k)}$ on V_k?

(Q2) Given a family of Markov chains $X^{(k)}$ on V_k, when does

$X^{(k)}$ converge weakly to a 'nice' Hunt process X as $k \to \infty$?

(Goal) To answer the questions when $X^{(k)}$ contains 'big jumps'.

Diffusion case **Case 0**: $\{\xi_i\}_i$: i.i.d., $X_t^{(k)} = \frac{1}{k}\sum_{i=1}^{[k^2 t]} \xi_i$. Then $X_t^{(k)} \to B_t$: BM.

Case 1: Non-divergence form (Stroock and Varadhan's book Chap. 11)

Weak convergence of some non-symmetric chains to a diffusion corresponding to

$$\mathcal{L}f(x) = \sum_{i,j=1}^{d} a_{ij}(x)\frac{\partial^2 f}{\partial x_i \partial x_j}(x) + \sum_{i=1}^{d} b_i(x)\frac{\partial f}{\partial x_i}(x).$$

$(a_{ij}(\cdot), b_i(\cdot)$: "regular enough") Key: Well-posedness of martingale problem.

Case 2: Divergence form (Stroock-Zheng ('97, Ann. IHP.))

Weak convergence of some symmetric chains to a diffusion corresponding to

$$\mathcal{L}f(x) = \sum_{i,j} \frac{\partial}{\partial x_i}(a_{ij}(x)\frac{\partial}{\partial x_j})f(x).$$

$(a_{ij}(\cdot)$: m'ble, unif. elliptic) Key: Apriori estimates for solutions of heat equations

(Discrete version of the De Giorgi-Moser-Nash theory.)

• Stroock-Zheng ('97, Ann. IHP.): Markov chain approximations to symmetric diffusions

$C^k(\cdot,\cdot) : \mathcal{S}_k \times \mathcal{S}_k \setminus \Delta \to \mathbb{R}_+$ conductance (i.e. $C^k(x,y) = C^k(y,x)$) satisfying

$$C^k(x,y) = 0 \;\text{ if }\; |x-y| \geq \exists\, R_0/k \quad (\text{finite range}), \qquad (1.1)$$

$$\lim_{k\to\infty} \sum_{l\in\mathcal{S}_k} \sup_{|y|\leq\forall r} \sup_{|x-y|\leq R_0/k} |C^k(x,x+l) - C^k(y,y+l)| = 0 \quad (\text{regularity}). \qquad (1.2)$$

$$C^k(x,y) \geq 1 \;\text{ if }\; |x-y| = 1/k, \qquad \sup_x \sum_{y\in\mathcal{S}_k} C^k(x,y) < \infty. \qquad (1.3)$$

Let $\{Y_t^{(k)}\}_t$: MC with transition prob. $\frac{C^k(x,y)}{\sum_y C^k(x,y)}$, exp. holding time (parameter k^{-2}).

$$a_k(x) := \sum_{e\in\mathbb{Z}^d} C^k(x,e/k)e \otimes e \quad \text{for } x \in \mathcal{S}_k,$$

and assume $\exists a : \mathbb{R}^d \to \mathbb{R}^d \otimes \mathbb{R}^d$ such that

$$\lim_{k\to\infty} \int_{|x|\le r} |a_k([x]_k) - a(x)| dx = 0 \quad \forall r > 0, \tag{1.4}$$

where $[x]_k = ([kx_1]/k, [kx_2]/k, \ldots, [kx_d]/k)$.

Obtain

(a) Weak convergence of $\{Y_t^{(k)}\}_t$ to a divergence form under (1.1)-(1.4), (Ans. (Q1))

(b) discrete approx. of divergence forms by symmetric MC's. (Ans. (Q2))

We consider the following cases:

- Limit process is a process associated with div. form + (possibly) jumps,
- Limit process is a pure jump process.

2 Divergence form+ jumps

- Bass-K ('08 TAMS): Instead of (1.1) and (1.2), assume

$$\sup_k \sup_{x\in V_k}\Big(k^2 \sum_{y\in V_k} |y|^2 C^{(k)}(x,x+y)\Big) < \infty,$$

and answer (Q2). (Rem: (Q1) can be done by S-Z.)

- Bass-K-Uemura ('10 PTRF): Instead of (1.1) and (1.2), assume

$$\sup_k \sup_{x\in V_k}\Big(k^2 \sum_{y\in V_k} (1\wedge |y|^2) C^{(k)}(x,x+y)\Big) < \infty,$$

and answer (Q2) [convergence to a divergence form+jump part].

Key: apriori estimates for solutions of heat equations

(Discrete version of the De Giorgi-Moser-Nash theory with jumps).

2.1 Tightness and heat kernel estimates

$C^k(\cdot,\cdot) : \mathcal{S}\times\mathcal{S}\setminus\Delta \to \mathbb{R}_+$ is a conductance ($C^k(x,y) = C^k(y,x)$). Assume

(A1) $\exists c_1, c_2 > 0$ *independent of k such that*

$$c_1 \le \nu_x^k := \sum_{y\in\mathcal{S}_k} C^k(x,y) \le c_2 \quad \text{for all} \quad x \in \mathcal{S}_k.$$

(A2) $\exists M_0 \ge 1, \delta > 0$ *(indep. of n) s.t. the following holds:*

$\forall x, y \in \mathcal{S}_k$ with $|x-y| = k^{-1}$, $\exists N \ge 2$ and $x_1, \cdots, x_N \in B_k(x, k^{-1}M_0)$ s.t.

$x_1 = x$, $x_N = y$ and $C^k(x_i, x_{i+1}) \ge \delta$ for $i = 1, \cdots, N-1$.

(A3) $\exists \varphi : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ *s.t.* $\forall k \in \mathbb{N}$,

$$C^k(x,y) \le k^{-(d+2)}\varphi\big(|x-y|\big), \quad x, y \in \mathcal{S}_k \quad \text{and} \quad \int_0^\infty \left(1\wedge t^2\right) t^{d-1}\varphi(t)dt < \infty.$$

Especially, (A3) implies

$$M := \sup_k \sup_{x\in\mathcal{S}_k}\Big(k^2 \sum_{y\in\mathcal{S}_k}(1\wedge|y|^2)C^k(x,x+y)\Big) < \infty. \tag{2.1}$$

Let $\mu_x^k := k^{-d}$, $\forall x \in \mathcal{S}_k$. $\mu^k(A) := \sum_{y\in A} \mu_y^k$, $\forall A \subset \mathcal{S}_k$. Define

$$\mathcal{E}^k(f,f) = \frac{k^{2-d}}{2} \sum_{x,y\in\mathcal{S}_k} (f(y) - f(x))^2 C^k(x,y), \quad \mathcal{F}^k = L^2(\mathcal{S}_k, \mu^k).$$

$Y_t^{(k)}$: corresponding continuous time Markov chain on $\mathcal{S}_k$

$\mathcal{A}^k$: infinitesimal generator of $Y_t^{(k)}$: $\mathcal{A}^k f(x) = \sum_{y\in\mathcal{S}_k} (f(y) - f(x)) C^k(x,y) k^2$

Proposition 2.1 $\forall A > 0$ *and* $0 < \forall B < 1$, $\exists t_0 = t_0(A,B) \in (0,1)$ *s.t.*

$$\mathbb{P}^x \left(\sup_{s \le r^2 t_0} |Y_t^{(k)} - Y_0^{(k)}| > rA \right) \le B, \qquad \forall k \in \mathbb{N}, r \in (0,1], x \in \mathcal{S}_k. \tag{2.2}$$

$Z^{(k)} = \{Z_t^{(k)} := (V_0 + t, Y_t^{(k)}), t \ge 0\}$: space-time process of $Y^{(k)}$.

Theorem 2.2 $\forall R_0 \in (0,1]$, $\exists c = c(R_0) > 0$, $\kappa > 0$ *s.t.* $0 < \forall R \le R_0$, $\forall k \ge 1$, *and*

$\forall q$: *bdd caloric in* $Q^k(0, x_0, 2R) := (0, 4R^2) \times B_k(x_0, 2R)$,

$$|q(s,x) - q(t,y)| \le c\, \|q\|_{\infty,R} \left(\frac{|t-s|^{1/2} + |x-y|}{R} \right)^{\kappa} \quad \forall (s,x),\ (t,y) \in Q^k(0, x_0, R),$$

where $\|q\|_{\infty,R} := \sup_{(t,y)\in[0,\,\gamma(4R)^2]\times\mathcal{S}_n} |q(t,y)|$.

In particular, for $p^k(t,x,y)$*: transition density function of* $Y^{(k)}$,

$$|p^k(s, x_1, y_1) - p^k(t, x_2, y_2)| \le c\, t_0^{-(d+\kappa)/2} \Big(|t-s|^{1/2} + |x_1 - x_2| + |y_1 - y_2| \Big)^{\kappa},$$

for $k^{-1} < \forall t_0 < 1$, $\forall t,\ s \in [t_0,\ 1]$ *and* $\forall (x_i, y_i) \in \mathcal{S}_k \times \mathcal{S}_k$ *with* $i = 1, 2$.

Cf. Z.-Q. Chen - K ('10 Rev. Mat. Iberoamericana)

By these results, we know that a subseq. of $\{Y_t^{(k)}\}_{0\le t\le t_0}$ converges weakly.

2.2 Weak convergence of Markov chains

Fix $\{\varepsilon_k\}$ such that $1 \geq \varepsilon_k \searrow 0$, and define

$$C_C^k(x,y) := C^k(x,y)1_{\{|x-y|\leq\varepsilon_k\}}, \quad C_J^k(x,y) := C^k(x,y) - C_C^k(x,y).$$

Now define 'cont. part' and 'jump part' of the D-form $(\mathcal{E}^k, \mathcal{F}^k)$; for $f \in L^2(\mathcal{S}_k, \mu^k)$,

$$\mathcal{E}_*^k(f,g) := \frac{k^{2-d}}{2} \sum_{x,y\in\mathcal{S}_k} \big(f(x) - f(y)\big)\big(g(x) - g(y)\big) C_*^k(x,y), \quad \text{where } * = C \text{ or } J.$$

Clearly $\mathcal{E}^k(f,g) = \mathcal{E}_C^k(f,g) + \mathcal{E}_J^k(f,g)$.

Set $G_{ij}^k(w,z)$ in such a way that

$$\mathcal{E}_C^k(u,v) = \frac{1}{2k^d} \sum_{i,j=1}^{d} \sum_{w,z\in\mathcal{S}_k} \nabla_{1/k}^i u(z) \nabla_{1/k}^j v(w) G_{ij}^k(w,z), \tag{2.3}$$

where $\nabla_h^i u(x) = (u(x + h\boldsymbol{e}_i) - u(x))/h$, and define $F_{ij}^k(z) = \sum_{w\in\mathcal{S}_k} G_{ij}^k(w,z)$.

We now give an assumption needed to obtain weak convergence of the processes.

(A4) $\exists\{\varepsilon_n\}$ *satisfying* $1/k \le \varepsilon_k \le 1$ *and* $\varepsilon_k \searrow 0$, *sym. matrix* $a(x) = (a_{ij}(x))$ *on* $\mathbb{R}^d$, *and sym. funct.* $j(x,y)$ *on* $\mathbb{R}^d \times \mathbb{R}^d \backslash D$ *s.t.* $\sup_{i,j,k} ||F_{ij}^k||_\infty < \infty$, $\forall B$: *compact,*

$$\int_B |F_{ij}^k(x) - a_{ij}(x)|\, dx \longrightarrow 0, \qquad \forall i,j = 1,2,\dots,d,$$

and

$$\lambda^{-1}|\xi|^2 \le \sum_{i,j=1}^d \xi_i \xi_j a_{ij}(x) \le \lambda|\xi|^2, \qquad x,\xi \in \mathbb{R}^d,$$

for some $\lambda > 0$. *Further, for each* $N > 1$, *the measures*

$$k^{d+2} C^k(x,y) \mathbf{1}_{[N^{-1},N]}(|x-y|) dx\, dy \overset{waekly}{\longrightarrow} j(x,y) \mathbf{1}_{[N^{-1},N]}(|x-y|) dx\, dy. \tag{2.4}$$

– Here we extend the domain of $C^k(\cdot,\cdot)$ from $\mathcal{S}_k \times \mathcal{S}_k$ to $\mathbb{R}^d \times \mathbb{R}^d$ in a natural way.

Since a is uniformly elliptic, if we define

$$\mathcal{E}(f,g) := \frac{1}{2}\int_{\mathbb{R}^d} \nabla f(x)\cdot a(x)\nabla g(x)dx + \frac{1}{2}\iint_{x\neq y}\big(f(x)-f(y)\big)\big(g(x)-g(y)\big)j(x,y)dx\,dy$$

then $(\mathcal{E}, C_c^1(\mathbb{R}^d))$ is a closable Markov. form on $L^2(\mathbb{R}^d, dx)$. Denote the closure by $(\mathcal{E}, \mathcal{F})$.

Theorem 2.3 *Suppose (A1)-(A4) hold.*

Then $\forall x$, $\forall t_0$, $\mathbb{P}^{[x]_k}$*-laws of* $\{Y_t^{(k)}\}_{0\le t\le t_0}$ *converge weakly on* $D([0,t_0],\mathbb{R}^d)$.

Let $\mathbb{P}^x$ *be the weak lim. and* Z_t *be the corresponding process on* $D([0,t_0),\mathbb{R}^d)$, *then* $(\mathcal{E}, W^{1,2}(\mathbb{R}^d))$ *is the D-form corresponding to* $\{Z_t, \mathbb{P}^x\}$.

How to define $G^k_{ij}(w,z)$? – Here is one way.

Let $L_{x,y}$ be the union of the line seg. from x to $(y_1, x_2, \cdots, x_d)$, the line seg. from $(y_1, x_2, \cdots, x_d)$ to $(y_1, y_2, x_3 \cdots, x_d)$, $\cdots$, and the line seg. from $(y_1, \cdots, y_{d-1}, x_d)$ to y, and write $L_{xy} = (p_1, p_2, \cdots, , p_N)$. Now define

$$P^{x,y}(w,z) := \begin{cases} 1, & \text{if } \exists \ell \text{ s.t. } w = p_\ell \text{ and } z = p_{\ell+1}, \\ 0, & \text{otherwise.} \end{cases}$$

For $i, j = 1, 2, \ldots, d$ and $w, z \in \mathcal{S}_k$, set

$$G^k_{ij}(w,z) \ := \sum_{x,y \in \mathcal{S}_k} \Big(P^{x,y}(z + \boldsymbol{e}_i/k, z) - P^{x,y}(z, z + \boldsymbol{e}_i/k) \Big) \times \Big(P^{x,y}(w + \boldsymbol{e}_j/k, w) - P^{x,y}(w, w + \boldsymbol{e}_j/k) \Big) C^k_C(x,y).$$

Then this $G^k_{ij}(w,z)$ satisfies (2.3).

3 Pure jumps (Chen-Kim-K '10)

$$\mathcal{E}(u,v) := \frac{1}{2}\int_{\mathbb{R}^d\times\mathbb{R}^d\setminus\Delta}(u(x)-u(y))(v(x)-v(y))j(x,y)dxdy.$$

((Heat kernel approaches to pure jump process))

- Husseini-Kassmann ('07 PA): Answer (Q1), (Q2) when $j(x,y) \asymp |x-y|^{-d-\alpha}$.
- Bass-Kassmann-K ('10 Ann. IHP):

  Answer (Q1), (Q2) for a class of jump proc. including

  $\frac{c_1}{|x-y|^{d+(s(x)\wedge s(y))}} \le j(x,y) \le \frac{c_2}{|x-y|^{d+(s(x)\vee s(y))}}$, where $|s(x)-s(y)| \le \frac{c}{\log(2/|x-y|)}$, $|x-y|<1$.

We cannot go very far!

- Barlow-Bass-Chen-Kassmann ('09 TAMS): Even in the case

$$\frac{c_1}{|x-y|^{d+\alpha_1}} \le j(x,y) \le \frac{c_2}{|x-y|^{d+\alpha_2}} \qquad \text{for } |x-y|<1, \quad 0<\alpha_1<\alpha_2<2,$$

$\exists$ an example where there is a bounded harmonic function that is **not** continuous.

3.1 Model and tightness

Let $V_k = k^{-1}\mathbb{Z}^d$, $m_k(x) = k^{-d}$ for all $x \in V_k$, and $B_j = B(0,j)$.

$\{\mathcal{C}^{(k)}(x,y),\ x,y \in V_k\}$: family of conductance

(i.e. $\mathcal{C}^{(k)}(x,y) = \mathcal{C}^{(k)}(y,x) \geq 0$. Assume $\mathcal{C}^{(k)}(x,x) = 0$.)

$(\mathcal{E}^{(k)}, \mathcal{F}^{(k)})$: Dirichlet form on $L^2(V_k, m_k)$ defined by

$$\mathcal{E}^{(k)}(u,v) := \frac{1}{2}\sum_{x,y\in V_k}(u(x)-u(y))(v(x)-v(y))\mathcal{C}^{(k)}(x,y)m_k(x)m_k(y) \quad \forall u,v \in \mathcal{F}^{(k)},$$

where $\mathcal{F}^{(k)} := \{u \in L^2(V_k; m_k)\,;\, \mathcal{E}^{(k)}(u,u) < \infty\}$.

(B1). $\exists k_0 \geq 1$ s.t. $\forall j \geq 1$,

$$\sup_{k\geq k_0}\sup_{x\in\overline{B}_j\cap V_k}\sum_{y\in V_k}\mathcal{C}^{(k)}(x,y)\Big(|x-y|^2 \wedge 1\Big)m_k(y) < \infty,$$

$$\sup_{k\geq k_0}\sup_{x\in B^c_{j+2}\cap V_k}\sum_{y\in B_j\cap V_k}\mathcal{C}^{(k)}(x,y)m_k(y) < \infty.$$

Under **(B1)**, $(\mathcal{E}^{(k)}, \mathcal{F}^{(k)})$ is regular.

Let $\left\{ \{X_t^{(k)}\}_{t \geq 0}, \{\mathbb{P}_x^{(k)}\}_{x \in V_k} \right\}$ be the corresponding symmetric MC.

$\forall \varphi \in C_c^+(\mathbb{R}^d)$, define

$$\mathbb{P}_\varphi^{(k)}(\,\cdot\,) := \sum_{x \in V_k} \mathbb{P}_x^{(k)}(\,\cdot\,)\varphi(x) m_k(x) \quad \text{and} \quad \mathbb{P}_\varphi(\,\cdot\,) := \int_{\mathbb{R}^d} \mathbb{P}_x(\,\cdot\,)\varphi(x) m(dx).$$

Let $\zeta^{(k)}$: the lifetime of the process $X^{(k)}$.

Proposition 3.1 *Assume* **(B1)** *holds. Then,* $\forall \varphi \in C_c^+(\mathbb{R}^d)$*, the laws of* $\{X_t^{(k)}\}_{t \in [0,T]}$ *on* $\{\zeta^{(k)} > T\}$ *with initial distri.* $\varphi(x) m_k(dx)$ *is tight in* $\mathbb{D}([0, \infty), \mathbb{R}^d)$.

Rem: We don't have tightness when the initial distribution is concentrated on a point.

3.2 Main theorem: weak convergence and discrete approximation

Define $\pi_k : L^2(\mathbb{R}^d, m) \to L^2(V_k, m_k)$ and $E_k : L^2(V_k, m_k) \to L^2(\mathbb{R}^d, m)$ as follows:

$$\pi_k f(x) = \frac{1}{m_k(x)} \int_{U_k(x)} f(y) m(dy), \qquad E_k g(z) = g(x) \ \text{ for } z \in U_k(x), \ \ x \in V_k.$$

Let $P_t f(x) := \mathbb{E}_x[f(X_t)]$ and $P_t^{(k)} g(x) := \mathbb{E}_x^{(k)}[g(X_t^{(k)})]$.

Theorem 3.2 *Assume that* **(B2)**-**(B4)** *hold.*

Then $E_k P_t^{(k)} \pi_k$ converges to P_t strongly (and uniformly for $t \le T$) in $L^2(M, m)$.

Theorem 3.3 *Assume* **(B1)**–**(B4)** *and that X is conservative. Then, for all $\varphi \in \mathrm{Lip}_c^+(\mathbb{R}^d)$, $\{(X^{(k)}, \mathbb{P}_\varphi^{(k)}); k \ge 1\}$ converges weakly to $(X, \mathbb{P}_\varphi)$.*

Remark: i) All the above results hold in a class of MMS with VD.

ii) We have another version of the results that does not require **(B3)**(ii).

(B2). $j : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}_+$ *is sym. m'ble s.t.*

$$\sup_{x \in K} \int_{\mathbb{R}^d} (|x-y| \wedge 1)^2 j(x,y) m(dy) < \infty, \quad \forall K : \ \textit{compact}. \tag{3.1}$$

$(\mathcal{E}, \mathcal{F})$: Dirichlet form on $L^2(\mathbb{R}^d, m)$ defined by

$$\mathcal{E}(u,v) := \frac{1}{2} \int_{\mathbb{R}^d \times \mathbb{R}^d \setminus \Delta} (u(x)-u(y))(v(x)-v(y)) j(x,y) m(dx) m(dy) \qquad \forall u, v \in \mathcal{F},$$

where $\mathcal{F} := \{u \in L^2(\mathbb{R}^d, m) : \mathcal{E}(u,u) < \infty\}$. Under (3.1), $\mathrm{Lip}_c(\mathbb{R}^d) \subset \mathcal{F}$.

(B3). (i) $\mathrm{Lip}_c(\mathbb{R}^d)$ is dense in $(\mathcal{F}, \mathcal{E}(\cdot,\cdot) + \|\cdot\|_2^2)$.

(ii) $\mathcal{L}_{j,\delta} f$ is continuous for all $f \in \mathrm{Lip}_c(\mathbb{R}^d)$ where $\mathcal{L}_{j,\delta}$ is given below.

Under **(B2)** and **(B3)**(i), $(\mathcal{E}, \mathcal{F})$ is regular.

Let $\{\{X_t\}_{t \geq 0}, \{\mathbb{P}_x\}_{x \in M}\}$ be the corresponding symmetric Hunt process.

We extend $\{\mathcal{C}^{(k)}(x,y) : x, y \in V_k\}$ to $\{\mathcal{C}^{(k)}(x,y) : x, y \in \mathbb{R}^d\}$ in a natural way. Set

$$\begin{aligned}
\overline{\mathcal{L}}^{(k)}_{j,\delta} u(x) &= \int_{B_j} (u(y) - u(x)) C^{(k)}(x,y) 1_{\{|x-y|>\delta\}} m(dy) \quad \forall x \in B_j, \\
\mathcal{L}_{j,\delta} u(x) &= \int_{B_j} (u(y) - u(x)) 1_{\{|x-y|>\delta\}} j(x,y) m(dy) \quad \forall x \in B_j.
\end{aligned}$$

(B4). (i) *For any compact subset* $K \subset \mathbb{R}^d$,

$$\lim_{\eta \to 0} \limsup_{k \to \infty} \int \int_{\{(x,y) \in K \times K : |x-y| \le \eta\}} |x-y|^2 \mathcal{C}^{(k)}(x,y) m(dx) m(dy) = 0, \tag{3.2}$$

$$\lim_{j \to \infty} \limsup_{k \to \infty} \int_K \int_{B_j^c} \mathcal{C}^{(k)}(x,y) m(dx) m(dy) = \lim_{j \to \infty} \sup_{x \in K} j(x, B_j^c) = 0. \tag{3.3}$$

(ii) $\lim_{k \to \infty} \|\overline{\mathcal{L}}^{(k)}_{j,\delta} f\|^2_{2,B_j} = \|\mathcal{L}_{j,\delta} f\|^2_{2,B_j}$, $\quad \forall f \in \mathrm{Lip}_c(\mathbb{R}^d)$, $\forall \delta > 0$ *and* $\forall j \in \mathbb{N}$.

(iii) $\mathcal{C}^{(k)}(x,y) m(dx) m(dy) \xrightarrow{waekly} j(x,y) m(dx) m(dy)$ *on* $B_j \times B_j \setminus \{|x-y| > \delta\}$.

Discrete approximation

Theorem 3.4 *Let* $j(x,y):\mathbb{R}^d\times\mathbb{R}^d\to\mathbb{R}_+$ *be s.t.*

$$j(x,y)1_{\{|x-y|\ge 1\}}\le M_0<\infty \quad \forall x,y\in\mathbb{R}^d, \qquad \lim_{j\to\infty}\sup_{x\in K} j(x,B_j^c)=0 \quad \forall K\subset\subset\mathbb{R}^d.$$

Assume that the Dirichlet form $(\mathcal{E},\mathcal{F})$ *determined by* $j(x,y)$ *satisfies* **(B2)**–**(B3)** *(i) and is conservative. Define*

$$\mathcal{C}^{(k)}(x,y):=\mathbf{1}_{\{|x-y|\ge c_1/k\}}\frac{1}{m_k(x)m_k(y)}\int_{U_k(x)}\int_{U_k(y)} j(\xi,\eta)m(d\xi)m(d\eta), \quad \forall x,y\in V_k.$$

Then, the corresponding $(\mathcal{E}^{(k)},\mathcal{F}^{(k)})$ *is a regular Dirichlet form on* $L^2(V_k,m_k)$*.*

Let $X^{(k)}$ *be its associated MC. Then, for all* $\varphi\in\mathrm{Lip}_c^+(\mathbb{R}^d)$,

$\{(X^{(2^k)},\,\mathbb{P}_\varphi^{(2^k)});\,k\ge 1\}$ *converges weakly to* $(X,\,\mathbb{P}_\varphi)$*.*

3.3 Application to Random Walk on Random Conductance

$\{\xi_{xy}\}_{x,y\in\mathbb{Z}^d, x\neq y}$: *i.i.d. on* $(\Omega, \mathcal{F}, \mathbf{P})$ *s.t.* $0 \le \xi_{xy}$, $M := E[\xi_{xy}] \in (0,\infty)$, *Var* $[\xi_{xy}] < \infty$.

Define random conductance $C(\cdot,\cdot)$ *as*

$$C(x,y) := \frac{\xi_{xy}}{|x-y|^{d+\alpha}}, \qquad d \ge 2,\ x,y \in \mathbb{Z}^d,\ 0<\alpha<\ 2.$$

$X^{(1)}$: *corresponding cont. time Markov chain on* $\mathbb{Z}^d$.

(I.e. $X^{(1)}$ *stays at* x *for an exponential length of time with parameter* $\sum_{z\neq x} C(x,z)$ *and then jumps to* y *w.p.* $C(x,y)/\sum_{z\neq x} C(x,z)$.*)*

$\Rightarrow$ $X^{(k)}_t = k^{-1} X^{(1)}_{k^\alpha t}$ *converges to* Z^α_{Mt} *in the f.d.d. sense* $\mathbf{P}$*-a.s. (quenched),*

where Z^α_t *is the (rotationally invariant)* α*-stable proc.*

- *Further, if* $\xi_{xy} \le C_1$, $\{(X^{(k)}, \mathbb{P}^{(k)}_\varphi); k \ge 1\}$ *converges weakly to* $(Z^\alpha_{M\cdot}, \mathbb{P}_\varphi)$ $\mathbf{P}$*-a.s.,*

  where $\mathbb{P}_\varphi(\cdot) := \int_{\mathbb{R}^d} \mathbb{P}_x(\cdot)\varphi(x)m(dx)$ *for* $\varphi \in C_c^+(\mathbb{R}^d)$.

⊙ $$C^{(k)}(x,y) := k^{d+\alpha}C(kx,ky) = \frac{\xi_{kx,ky}}{|x-y|^{d+\alpha}} \quad \textit{for} \quad x,y \in V_k.$$

Then, by LLN,

$$\int_{B(x_0,h)}\int_{B(y_0,h)} C^{(k)}(x,y)m(dx)m(dy) \sim \frac{\xi_1^{(k)} + \cdots + \xi_{2(hk)^d}^{(k)}}{k^{2d}|x_0-y_0|^{d+\alpha}} \to \frac{Ch^{2d}}{|x_0-y_0|^{d+\alpha}} \ a.s..$$

Taking $j(x,y) = M|x-y|^{-d-\alpha}$, **(B2)**–**(B4)** *hold* $\mathbf{P}$-*a.s., so obtain f.d.d. convergence.*

Further, if $\xi_{xy} \leq C_1$, *then* **(B1)**–**(B4)** *hold* $\mathbf{P}$-*a.s., so obtain weak convergence.*

Sketch of the proof of Proposition 3.1.

By the Lyons-Zheng decomposition, $\exists$ *a martingale* $M^{k,f}$ *s.t. on* $\{\zeta^{(k)} > T\}$,

$$f(X_t^{(k)}) - f(X_0^{(k)}) = \frac{1}{2}M_t^{k,f} - \frac{1}{2}(M_1^{k,f} - M_{(1-t)-}^{k,f}) \circ r_1, \quad t \in [0,1] \tag{3.4}$$

where $r_1(\omega)(s) = \omega((1-s)_- \vee 0)$. *Note that* $\forall f \in \mathrm{Lip}_c(\mathbb{R}^d)$ *and* $s < t$,

$$\begin{aligned} \langle M^{k,f}\rangle_t - \langle M^{k,f}\rangle_s &= \int_s^t \sum_{y \in V_k} (f(X_u^{(k)}) - f(y))^2 \mathcal{C}^{(k)}(X_u^{(k)}, y) m_k(y) du \\ &\le c(t-s) \qquad\qquad (by\ \mathbf{(B1)}) \end{aligned}$$

$\Rightarrow \{\langle M^{k,f}\rangle_t\}_{k\ge 1}$ *is* C*-tight in* $\mathbb{D}_{\mathbb{R}^d}[0,1]$ *(with the Skorokhod top.)*

$\Rightarrow$ *Since* $m_k \to m$, *laws of* $\{M^{k,f}\}_{k\ge 1}$ *is tight in* $\mathbb{D}_{\mathbb{R}^d}[0,1]$ *with the initial distri.* $\mathbb{P}_\varphi^{(k)}$.

Theorem 3.2 is equivalent to the following:

Theorem 3.5 $(\mathcal{E}^{(k)}, \mathcal{F}^{(k)})$ *is Mosco convergent to* $(\mathcal{E}, \mathcal{F})$ *in the generalized sense:*

(I) If $v_k \in L^2(V_k)$, $u \in L^2(\mathbb{R}^d)$ *and* $E_k v_k \to u$ *weakly in* $L^2(\mathbb{R}^d)$, *then*

$$\liminf_{k\to\infty} \mathcal{E}^{(k)}(v_k, v_k) \geq \mathcal{E}(u, u).$$

(II) For every $u \in L^2(\mathbb{R}^d)$, *there exists* $u_k \in L^2(V_k)$ *such that* $\|u_k\|_{k,2} \to \|u\|_2$, $(E_k u_k, f) \to (u, f)$ *for every* $f \in L^2(\mathbb{R}^d)$ *and*

$$\limsup_{k\to\infty} \mathcal{E}^{(k)}(u_k, u_k) \leq \mathcal{E}(u, u).$$

We use **(B2)–(B4)** *to prove this.*