Anomalous Behavior of Random Walks on Disordered Media

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Abstract
Much effort has been expended on investigation of the physical properties of disordered media (complex systems) including how the heat transfers on the media. In mathematics, these properties have been actively studied for over 30 years. In particular, probabilistic methods have been developed extensively to analyze random walks and their scaling limits on the media. This chapter provides a discussion of the behavior of random walks and diffusions on typical disordered media.

1 Introduction

Around the mid-1960s, mathematical physicists started investigating the anomalous behavior of heat transfer on disordered media (e.g., see Ben-Avraham and Havlin 2000). Examples of disordered media include polymers, complex networks, and growth of mold and crystal. In this chapter, we consider "disordered media" as a subclass of "complex systems." Mathematical progress on these problems started in the late 1980s. The first systematical progress was made on fractals, which are in some sense ideal disordered media because they have exact self-similarity. Meanwhile, analytical methods and techniques were gradually developed that enabled us to analyze quantitative estimates for heat transfer on some random media that had higher complexity. Probabilistic approaches to the problems are used to investigate random walks (RWs) and diffusions. Because one cannot expect smoothness on the objects, it is not easy to construct differential operators directly. Probability theory does not require smoothness of the objects, and it works well to analyze them.

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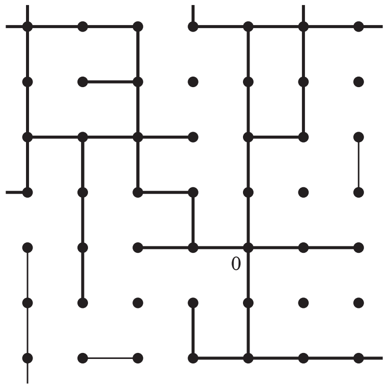

Fig. 1 Example of percolation cluster.

In this survey, we discuss behavior of random walks and diffusions on fractals and random media. In particular, we focus on the three random media presented in the following sections.

1.1 Bond Percolation on the Lattice

The first model is the bond percolation on d-dimensional lattice $\mathbb{Z}^d$, where $\mathbb{Z}$ is the set of integers. Let $d \geq 2$. On each bond with length 1, we flip a coin and open (resp. close) the bond if it lands on heads (resp. tails). Let $p \in [0, 1]$ be the probability that the coin lands on heads. (If $p \neq 1/2$, it is not a fair-coin.) We assume that flipping each coin is independent of flipping other coins. When all the coins are flipped, we have a set of open bonds; this model is called bond percolation. Let $C(0)$ be the set of vertices in $\mathbb{Z}^d$ that is connected to the origin by open bonds and let $\theta(p)$ be the probability that the set $C(0)$ is an infinite set. Then, it is known that this model enjoys phase transition in the following sense: there exists $p_c \in (0, 1)$ such that $\theta(p) = 0$ if $p < p_c$ and $\theta(p) > 0$ if $p > p_c$. The percolation model is one of the most fundamental models in statistical physics that has phase transitions. Our interest is in how heat transfers on random media.

1.2 The Erdős-Rényi Random Graph

The second model is the so-called Erdős-Rényi random graph, which is a standard model in the field of disordered networks. Let $N \geq 2$ be a natural number and set $V_N := \{1, 2, \cdots, N\}$. For each pair of distinct points $i, j \in V_N$, $i \neq j$, we connect the bond $\{i, j\}$ with probability $p \in [0, 1]$ and disconnect it with probability $1 - p$. As before, whether each bond is connected is independent of the situations of other

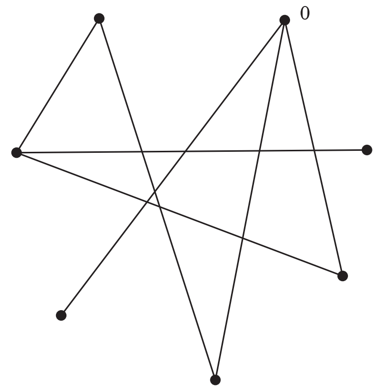

Fig. 2 Example of Erdős-Rényi random graph.

bonds. The resulting random graph is called the Erdős-Rényi random graph. When $p = 1$, it is the complete graph with vertices V_N, so the Erdős-Rényi random graph is the bond percolation for the complete graph on V_N. Let C^N be the largest connected component of the graph. It is known that this model enjoys sharp phase transition around $p = c/N$ with $c = 1$. Namely, the following holds with high probability:

$$\begin{aligned} c < 1 &\Longrightarrow |C^N| = O(\log N), \\ c > 1 &\Longrightarrow |C^N| \asymp N, \\ c = 1 &\Longrightarrow |C^N| \asymp N^{\frac{2}{3}}. \end{aligned}$$

Here $|A|$ is the number of the element in A, and we write $f(N) \asymp g(N)$ if there exist $c_1, c_2 > 0$ such that $c_1 f(N) \leq g(N) \leq c_2 f(N)$ for all N.

1.3 Two-Dimensional Uniform Spanning Tree

The third example of random media is the two-dimensional uniform spanning tree (2-Dim UST). Let $\Lambda_N := [-N, N]^2 \cap \mathbb{Z}^2$ and consider the graph that connects each neighboring bond with length 1. A loopless connected subgraph whose vertices consist of all the elements of Λ_N is called a spanning tree. Let $\mathcal{U}^{(N)}$ be a random graph that picks up one among all the spanning trees on Λ_N uniformly at random. The uniform spanning tree $\mathcal{U}$ is the limit of $\mathcal{U}^{(N)}$ as $N \to \infty$. This model is extremely important in modern probability theory. Some readers may have heard of the Schramm-Loewner evolution (SLE). It is a stochastic process heavily related to the works of two Fields medalists, W. Werner and S. Smirnov. 2-Dim UST is the model that O. Schramm, who invented the SLE, studied the scaling limit in his celebrated paper in 2000 that introduced SLE for the first time.

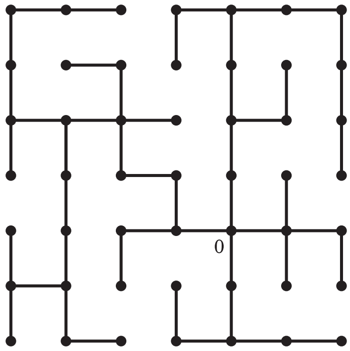

Fig. 3 Example of uniform spanning tree.

2 RW on the Lattice and Brownian Motion on $\mathbb{R}^d$

Before explaining RW on random media (random graphs), let us first explain simple random walk (SRW) on the d-dimensional square lattice $\mathbb{Z}^d$ and Brownian motion on the Euclidean space $\mathbb{R}^d$ that appears as a scaling limit of the SRW. Let $Y = \{Y_n\}_{n\in\mathbb{N}}$ be the SRW on $\mathbb{Z}^d$, namely it is a random motion such that for $x, y \in \mathbb{Z}^d$ with $|x - y| = 1$,

$$P(Y_{n+1} = y | Y_n = x) = \frac{1}{2d}.$$

In other words, it is a random motion of a particle that jumps to one of the nearest neighborhoods with equal probability.

Let us consider the scaling limit of the SRW by taking the mesh size of the lattice smaller and smaller. The geometric picture is that we take the limit $\varepsilon \to 0$ of $\varepsilon\mathbb{Z}^d$ so the spatial scaling limit is $\mathbb{R}^d$. Now let us consider the SRW εY_n on $\varepsilon\mathbb{Z}^d$. If we merely take $\varepsilon \to 0$, then the limiting process does not move at all, so we should speed up the time n depending on ε. It is known that we have the nontrivial (nondegenerate) limit process if we speed up the time by multiplying ε^{-2}, namely

$$\lim_{\varepsilon\to 0} \varepsilon Y_{[\frac{t}{\varepsilon^2}]} = B_t$$

and the limit process $\{B_t\}_{t\geq 0}$ is called Brownian motion, which is a random motion of a particle on $\mathbb{R}^d$. Brownian motion is related to the heat transfer on $\mathbb{R}^d$ because the differential operator (to be precise, the generator of the semigroup) determined by Brownian motion is

$$\frac{1}{2}\Delta := \frac{1}{2}\sum_{i=1}^{d} \frac{\partial^2}{\partial x_i^2},$$

that is $1/2$ times of the Laplace operator on $\mathbb{R}^d$. In fact, for a bounded continuous function f on $\mathbb{R}^d$, define $u(t, x) = E[f(B_t)|B_0 = x]$ (where E is the average with respect to Brownian motion). Then this $u(t, x)$ is the solution to the heat equation

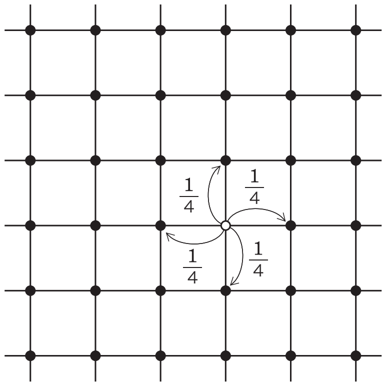

Fig. 4 Transition probability of two-dimensional simple random walk.

$$\frac{\partial u}{\partial t} = \frac{1}{2}\Delta u, \qquad \lim_{t\to 0} u(t,x) = f(x).$$

The heat kernel (fundamental solution to the heat equation) is the following Gauss kernel:

$$p_t(x,y) = \frac{1}{(2\pi t)^{\frac{d}{2}}} \exp\Big(-\frac{|x-y|^2}{2t}\Big).$$

We note that the time scale factor ε^{-2} for the SRW to have the scaling limit (which is Brownian motion) is related to the fact that the Laplace operator is the second-order differential operator.

3 RW on Fractal Graphs and Brownian Motion on Fractals

We next consider SRWs on the fractal graphs and their scaling limits. As a typical fractal, we consider the Sierpinski gasket, which is shown on the left of Fig. 5. Note that a standard Sierpinski gasket is a compact one, say K. We extend it to an unbounded one by letting the left bottom vertex of the triangle as the origin and define $\hat{K} = \cup_{m=0}^{\infty} 2^m K$. Let G be the Sierpinski gasket graph as shown on the right of Fig. 5, where the length of each bond is 1. Now let $Y = \{Y_n\}_{n\in\mathbb{N}}$ be the SRW on G, namely the particle jumps at one of the neighboring points (which is a point that is connected by a bond) with equal probability after 1 second. Let us consider the SRW $2^{-m}Y_n$ on $2^{-m}G$. As before, if we merely take $m \to \infty$, then the limiting process does not move at all, so we should speed up the time n depending on m. It turns out that if we speed up the time by multiplying 5^m, then we have the nontrivial (nondegenerate) limit process, namely

$$\lim_{m\to\infty} 2^{-m} Y_{[5^m t]} = B_t,$$

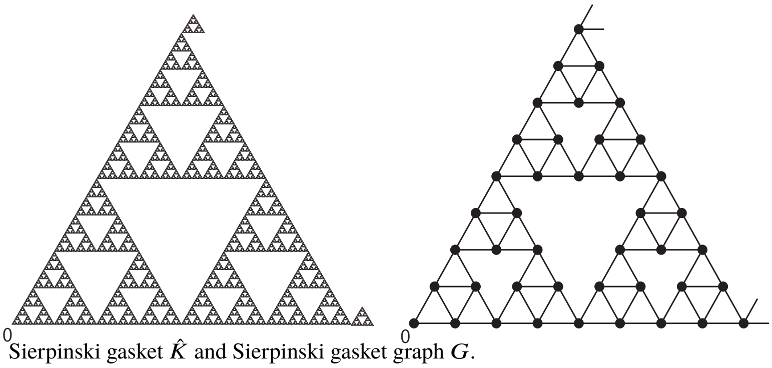

Fig. 5 Sierpinski gasket $\hat{K}$ and Sierpinski gasket graph G.

and the limit process $\{B_t\}_{t\geq 0}$ on the gasket is called Brownian motion, which is a random motion of a particle on $\hat{K}$. Brownian motion on the gasket was first constructed by Goldstein (1987) and Kusuoka (1987) independently.

The Laplace operator $\mathcal{L}$ that corresponds to Brownian motion was first constructed by Kigami (1989) and it can be determined as follows:

$$\mathcal{L}f(x) = \lim_{m\to\infty} 5^m \Big(\sum_{x_i : x \overset{m}{\sim} x_i} f(x_i) - 4f(x) \Big), \quad x \in \cup_{m\geq 0} 2^{-m} G \setminus \{0\}.$$

Here $x \overset{m}{\sim} y$ means that x and y are neighborhood on $2^{-m}G$. We note that the classical Laplace operator on $\mathbb{R}$ can be written as $\Delta f(x) = \lim_{m\to\infty} 2^{2m}(f(x+2^{-m}) + f(x - 2^{-m}) - 2f(x))$ for $f \in C^2(\mathbb{R})$. Let $d_w = \log 5/\log 2$ (hence $5 = 2^{d_w}$). Naively, we can say that the Laplacian on the gasket is a "differential operator of order d_w". (One mathematical justification of this is that the domain of the so-called Dirichlet form on the gasket is a Besov space of order $d_w/2$.)

We can consider a d-dimensional gasket in a similar way in $\mathbb{R}^d$ from the family of $(d+1)$-th contraction maps with contraction rate $1/2$. (For $d = 1$, $\hat{K} = [0, \infty)$.) The Hausdorff (fractal) dimension and the walk dimension of the d-dimensional gasket are $d_f = \log(d+1)/\log 2$ and $d_w = \log(d+3)/\log 2$, respectively.

Let $d(x, y)$ be the shortest distance between x and y in $\hat{K}$. It is known that there exists a heat kernel (fundamental solution of the heat equation) $p_t(\cdot,\cdot)$ and the following sub-Gaussian heat kernel estimates holds for all $t > 0, x, y \in \hat{K}$ (Barlow-Perkins 1988):

$$c_1 t^{-\frac{d_f}{d_w}} \exp\Big(- c_2 \Big(\frac{d(x,y)^{d_w}}{t} \Big)^{\frac{1}{d_w - 1}} \Big) \leq p_t(x,y) \leq c_3 t^{-\frac{d_f}{d_w}} \exp\Big(- c_4 \Big(\frac{d(x,y)^{d_w}}{t} \Big)^{\frac{1}{d_w - 1}} \Big). \quad (1)$$

The simple random walk on G also enjoys (1) for $d(x, y) \leq t \in \mathbb{N}$ (Jones 1996). Recall that for Brownian motion on $\mathbb{R}^d$, $p_t(x, y) = \frac{1}{(2\pi t)^{d/2}} \exp(-|x-y|^2/(2t))$. On fractals, we do not have explicit equality, but a generalized version of the heat kernel estimates hold, which is already good enough to deduce various analytical properties of the process. Note that when $d_f = d$ and $d_w = 2$, (1) boils down to the Gaussian estimates.

d_w is heavily related to properties of Brownian motion on the gasket. Indeed, by integrating (1), we have $c_5 t^{1/d_w} \le E^x[d(x, B(t))] \le c_6 t^{1/d_w}$; that is, d_w is the order of the average diffusion speed of particles. Given that $d_w > 2$, the behavior of the process is anomalous (for a long time, it diffuses slower than Brownian motion on $\mathbb{R}^d$, so the behavior is sub-diffusive). Set $d_s/2 = d_f/d_w$. d_s, which will appear in (2) again, gives the asymptotic growth of the eigenvalue counting function for Laplacian on the compact gasket K, and it is called the spectral dimension. In analysis, it is extremely important to analyze spectral properties of the Laplacian; on fractals, these properties have been extensively studied since early 1990s (Fukushima-Shima 1992 etc.).

There are many other fractals on which natural diffusion processes are constructed and studied; for instance, on nested fractals and p.c.f. self-similar sets, and also on Sierpinski carpets. It turns out that the theory of Dirichlet forms is applicable to this area. For example, see Barlow (1998) for details.

4 SRW on the Percolation Cluster

In the following three sections, we discuss RWs on random media and their scaling limits. From now on, the space we consider is always random (note that there are two randomnesses, one is that of the space and the other is that of the RW). Let us write ω for the randomness of the media. That is, the random objects we denoted by $\mathcal{C}$ and $\mathcal{U}$ in Section 1 will be denoted by $\mathcal{C}(\omega)$ and $\mathcal{U}(\omega)$ when we take one realization of the random graph.

In this section, we discuss the percolation cluster. Denote the SRW on the percolation cluster by $Y = \{Y_n^\omega\}_{n\in\mathbb{N}}$. Namely, Y_n^ω is located on one of the neighborhoods of Y_{n-1}^ω and it is equally distributed among all the neighborhoods. As mentioned above, ω stands for the randomness of the media; we fix $\mathcal{C}(0) = \mathcal{C}(0)(\omega)$ and consider SRW on it. SRW on the percolation cluster is sometimes called "the ant in the labyrinth."

4.1 Supercritical Case

We first consider the supercritical case, that is, when $p > p_c$. In this case, it is known that there is a unique infinite open cluster. In the following, we condition on the case $|\mathcal{C}(0)| = \infty$. Then, the SRW on the cluster enjoys similar long-time behavior as that of the SRW on $\mathbb{Z}^d$ although there are many holes on the media. Indeed, it is known that this SRW enjoys the following Gaussian heat kernel estimates for large n almost surely with respect to the randomness of the media (Barlow 2004):

$$c_1 n^{-\frac{d}{2}} \exp\left(-c_2 \frac{|x-y|^2}{n}\right) \le p_n^\omega(x,y) + p_{n+1}^\omega(x,y) \le c_3 n^{-\frac{d}{2}} \exp\left(-c_4 \frac{|x-y|^2}{n}\right).$$

Furthermore, the scaling limit of the SRW is similar to that of the SRW on $\mathbb{Z}^d$. That is, there exists a (nonrandom) constant $\sigma > 0$ such that the following holds almost surely with respect to the randomness of the media (Sidoravicius-Sznitman 2004; Berger-Biskup 2007; Mathieu-Piatnitski 2007):

$$\lim_{\varepsilon \to 0} \varepsilon Y^\omega_{[\frac{t}{\varepsilon^2}]} = \sigma B_t.$$

As we see, for the supercritical case, although there are many holes in the media, the long-time behavior of the SRW is similar to that of the SRW without holes. [In fact, if we search more detailed properties of the SRW, we can find differences between the two SRWs. We omit details and refer to Biskup (2011) and Kumagai (2014).]

4.2 Critical Case

Alexander and Orbach (1982) conjectured that the behavior of SRW on the critical percolation is completely different from that of SRW on $\mathbb{Z}^d$. As before, let $p_n^\omega(x,y)$ the heat kernel for the SRW. We call the following quantity (if the limit exists) spectral dimension:

$$d_s := -2 \lim_{n\to\infty} \frac{\log p_{2n}^\omega(x,x)}{\log n}. \quad (2)$$

One mathematical formulation of the Alexander-Orbach conjecture is that the spectral dimension for the SRW on the critical percolation is $4/3$ regardless of the dimension $d \ge 2$. For SRW on $\mathbb{Z}^d$, it holds that $d_s = d$, so this conjecture says the SRW on the critical percolation is anomalous like those on fractals.

To tackle this conjecture, the first problem is that there is no infinite cluster at $p = p_c$ (that is $\theta(p_c) = 0$) at least for $d = 2$ and $d \ge 11$. [In fact, it is a major open problem in this area whether $\theta(p_c) = 0$ for all $d \ge 2$ or not. It is believed that it is the case.] It is known that RWs on finite graphs converge to the stationary state under very mild conditions, so the limit in (2) will be 0, which is not what we want. So, we consider the so-called incipient infinite cluster (IIC), which is defined as follows. Consider the conditional probability that $C(0)$ intersects with the boundaries of the box of length N centered at 0 and then take $N \to \infty$; IIC is the (unique) infinite cluster on the probability space. It is known that at $p = p_c$, with high probability there is an open cluster with length of order n in the box of size n. So one can naturally believe that the mesoscopic behavior for the RW on the large finite cluster is similar to the long-time behavior of the RW on the IIC.

In general, analysis at critical probability is very difficult. So far, the IIC is rigorously constructed only for $d = 2$ and $d \ge 19$. (The former uses planar properties of $d = 2$ and the latter uses the renormalization technique at criticality called the lace expansion.) Let $Y = \{Y_n^\omega\}_{n\in\mathbb{N}}$ be the SRW on IIC and $p_n^\omega(x,y)$ be its heat kernel. Recently (2) has been proved with $d_s = 4/3$ almost surely with respect to the randomness of the media when the dimension is very high, namely the Alexander-Orbach conjecture is proved affirmatively in this case (Kozma-Nachmias 2009; cf. Barlow-Járai-Kumagai-Slade 2008). It was also revealed that the number 4/3 comes from the Hausdorff dimension of IIC $d_f = 2$ and the walk dimension $d_w = 3$ via a formula $d_s = 2d_f/d_w$. It is conjectured that the Alexander-Orbach conjecture does not hold for $d \le 5$, but there is no mathematically rigorous proof. Of note, disproving the conjecture for $d = 2$ is one of the challenging open problems in this area.

5 SRW on the Erdős-Rényi Random Graph

As observed in Section 1.2, the Erdős-Rényi random graph enjoys phase transition around $p = 1/N$. Here we fix $\lambda \in \mathbb{R}$, choose $p = N^{-1} + \lambda N^{-4/3}$, and study the spatial scaling limit at the critical window. When p is in this critical window, it is known that $|C^N| \asymp N^{2/3}$ (Aldous 1997).

Let us first explain the geometrical scaling limit of the random graph. We regard C^N as a metric space with origin. Then it is proved that when $N \to \infty$, there exists a random compact set $\mathcal{M} = \mathcal{M}_\lambda$ such that the following holds,

$$N^{-\frac{1}{3}} C^N \longrightarrow \mathcal{M},$$

(Addario-Berry, Broutin, Goldschmidt 2012). Here the convergence is in the sense of Gromov-Hausdorff, but we omit details. Let $\{Y_n^{C^N}\}_{n\ge 0}$ be the SRW on C^N. Then the following holds,

$$\lim_{N\to\infty} N^{-\frac{1}{3}} Y^{C^N}_{[Nt]} = B_t^{\mathcal{M}},$$

(Croydon 2012). Here $\{B_t^{\mathcal{M}}\}_{t\ge 0}$ is the Brownian motion on $\mathcal{M}$. Furthermore, there exists the heat kernel $p_t^{\mathcal{M}}(\cdot,\cdot)$ for Brownian motion such that the following estimates hold for all $x, y \in \mathcal{M}$ and $t \le 1$ (Croydon 2012),

$$p_t^{\mathcal{M}}(x,y) \le c_1 t^{-\frac{d_f}{d_w}} \ell(t^{-1})^{\theta} \exp\left\{-c_2 \left(\frac{d(x,y)^{d_w}}{t}\right)^{\frac{1}{d_w-1}} \ell\left(\frac{d(x,y)}{t}\right)^{-\theta}\right\}, \tag{3}$$

$$p_t^{\mathcal{M}}(x,y) \ge c_3 t^{-\frac{d_f}{d_w}} \ell(t^{-1})^{-\theta} \exp\left\{-c_4 \left(\frac{d(x,y)^{d_w}}{t}\right)^{\frac{1}{d_w-1}} \ell\left(\frac{d(x,y)}{t}\right)^{\theta}\right\}. \tag{4}$$

Here $\theta > 0$, $\ell(x) := 1 \vee \log x$, $d_f = 2, d_w = d_f + 1 = 3$ and $d(\cdot,\cdot)$ is the metric naturally defined on $\mathcal{M}$. As discussed above, Brownian motion on fractals enjoy similar sub-Gaussian heat kernel estimates with $\ell(x) = 1$. SRWs around critical probability and their scaling limits enjoy heat kernel estimates similar to the sub-

Gaussian estimates, and because of the randomness of the media, there is oscillation of the logarithmic order.

6 SRW on the 2-Dim UST

Let us first explain the geometrical scaling limit of the 2-Dim UST. In the paper of Schramm (2000), topological properties of a candidate of a scaling limit of the UST were analyzed. The space is $\mathbb{R}^2$ as a set, but the topological structure of the space is given by the embedding of some tree into $\mathbb{R}^2$, and it is very different from the one we usually consider using the Euclidean metric. Later, Lawler-Schramm-Werner proved that the scaling limit exists uniquely. UST can be constructed as a collection of some random paths called the loop-erased RW, and it is known that the scaling limit of the loop-erased RW is SLE_2, which is in the class of the Schramm-Loewner evolutions. The scaling limit of the 2-Dim UST is thus heavily related to the theory of SLE.

We next discuss SRW on the 2-Dim UST. As before, we regard $\mathcal{U}$ as a metric space with origin and let $X^{\mathcal{U}}$ be the SRW on $\mathcal{U}$ starting at 0. Then the following holds (Barlow-Croydon-Kumagai 2017; Holden-Sun 2018):

$$\lim_{\varepsilon\to 0} \varepsilon X^{\mathcal{U}}_{\varepsilon^{-\frac{13}{4}}t} = Y_t. \tag{5}$$

Here $\{Y_t\}_{t\geq 0}$ is a stochastic process on $\mathbb{R}^2$, but it is completely different from Brownian motion on $\mathbb{R}^2$. Indeed, there exists the heat kernel of $\{Y_t\}_{t\geq 0}$ such that (3) and (4) hold with $\ell(x) := 1 \vee \log x$ and $d_f = 8/5, d_w = d_f + 1 = 13/5$. Here $d(\cdot,\cdot)$ is the metric on the tree that embeds into $\mathbb{R}^2$, and it is completely different from the Euclidean metric. $d_f = 8/5$ is the Hausdorff dimension of $\mathbb{R}^2$ with respect to the metric $d(\cdot,\cdot)$ (note that if we use the Euclidean metric, then the dimension of $\mathbb{R}^2$ is clearly 2), hence if we observe the exponents with respect to the Euclidean metric, then it should be multiplied by $5/4$. Indeed, the exponent $13/4 = (5/4) \cdot d_w$ appearing in (5) is the walk dimension of the process with respect to the Euclidean metric.

7 Conclusions

As observed in several concrete examples, SRWs on disordered random media and their scaling limits enjoy similar properties as those on fractals, which are anomalous and quite different from those on $\mathbb{Z}^d$ or on $\mathbb{R}^d$. These examples may have various applications such as dynamics on the Internet (for instance, how the viruses spread out on the Internet), and various other open problems. The interested reader may refer to Kumagai (2014) and references therein.

Acknowledgment This research was partly supported by JSPS KAKENHI Grant Number JP17H01093.

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Biography

Takashi Kumagai is a professor of mathematics at the Research Institute for Mathematical Sciences (RIMS) at Kyoto University in Japan. His research focuses on anomalous diffusions on disordered media, such as fractals and random media. Kumagai completed his Ph.D. at Kyoto University in 1994, and after working at Osaka University and Nagoya University, he accepted a position at Kyoto University in 1998. Kumagai was an invited speaker at the 2014 ICM in Seoul, and gave a Medallion Lecture at the Conference on Stochastic Processes and their Applications in Moscow in 2017. His awards include the Spring Prize of the Mathematical Society of Japan (2004), JSPS Prize (2012), Inoue Prize for Science (2017), Osaka Science Prize (2017) and Humboldt Research Award (2017).