Lectures on Spin Glasses

Erwin Bolthausen* Kyoto University and University of Zürich

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1 Preliminary topics

1.1 Overview

Spin glass theory is closely related to the problem to determine properties of the law of the maximum $\max_{\sigma\in\Sigma} H(\sigma)$ for stochastic random fields, where Σ is a finite (but large) set and $\mathbf{H} = \{H(\sigma)\}_{\sigma\in\Sigma}$ is a family of real valued random variables.

To write H for the random variables is done in order to align (partly) with the physicists viewpoint where H is a random Hamiltonian. However, physicists prefer to look at the minimum, the so-called ground state, but that's only a question of changing the sign, of course. We stay here with the probabilists preference to be interested in the maximum.

In many cases $\mathbf{H}$ is a Gaussian family. This simplifies quite a number of arguments.

It is important that there is a "soft" version of this problem, by introducing a parameter $\beta > 0$, the so-called inverse temperature, and investigating (in physicists jargon) the partition function

$$Z_\beta := \sum_\sigma \mathrm{e}^{\beta H(\sigma)}.$$

Physicists prefer to have a minus sign in the exponent, but this is a bit of a nuisance, and we stay with the $+$. It is clear that for large β, this sum is dominated by the summands where $H(\sigma)$ is near its maximum. The maximum can typically be recovered by a $\beta \to \infty$ procedure, but the analysis of the situation for finite (or even small) β is often *much* simpler.[1]

The main fascination is coming from the fact that there is a supposed universality for a large class of very different models, where for large Σ a limit object is appearing, Ruelle's probability cascades, introduced in [39] which are closely related to what is now called the Bolthausen-Sznitman [10] coalescent process. This is proved just in a very limited number of cases. Here are very few examples:

[1] Please keep in mind that β is the *inverse* temperature. Therefore, the case of small β, which is typically the less interesting situation, is called "high-temperature" case in physics jargon, which gives it a more interesting touch.

- The d-dimensional lattice free field: Here $\Sigma = \Sigma_N = \{1, \ldots, N\}^d$, and $\mathbf{H}$ is centered Gaussian with $\mathbb{E}H(\sigma) H(\sigma') = G_{N,d}(\sigma, \sigma')$, $G_{N,d}$ being the Green's function on Σ_N of the discrete Laplacian with Dirichlet boundary conditions. Probabilistically, $G_{N,d}(\sigma, \sigma')$ is the mean number of visits of σ' of a random walk starting in σ' with killing when exiting Σ_N. For $d = 1$, this is a random walk with Gaussian increments, tied down at both ends. This case is rather simple, and is not really related to spin glass theory. The most interesting case is $d = 2$ where $\max_\sigma H(\sigma)$ was first determined in leading order in Bolthausen-Deuschel-Giacomin [12], and where the relations with spin glass theory had been proved. Although it is a kind of "trivial" from the spin glass viewpoint, it got tremendous popularity recently, as it is related to branching random walks, SLEs, Liouville quantum gravity, and even the Riemann hypothesis, and there are many more refined works on it, e.g. by Bramson-Zeitouni [19], Biskup-Louidor [9], Arguin-Zindy [6], and many others.

- The Sherrington-Kirkpatrick model with $\Sigma_N = \{-1, 1\}^N$ and

$$H(\sigma) := \sum_{1 \le i < j \le N} g_{ij} \sigma_i \sigma_j$$

with g_{ij} being standard Gaussians (typically scaled down by a factor $\sqrt{N}$ which is not important for the moment). Here, the leading order of $\max_\sigma H(\sigma)$ is *much* more interesting than in the free field case, but also *much* less is known rigorously. The leading order was first rigorously determined by Talagrand [41], proving a famous formula of Parisi. There is a recent new approach on the topic by Panchenko [37], [38].

1.2 Notations

We generally use $(\Omega, \mathcal{F}, \mathbb{P})$ for the basic probability space on which the random Hamiltonians $H(\sigma)$ are defined. Occasionally, if we want to stress the random character, we write $H_\omega(\sigma)$. For a finite parameter $\beta > 0$, the **random Gibbs distribution** on Σ is defined by $\mathcal{G}_{\omega,\beta}$:

$$\mathcal{G}_{\beta,\omega}(\sigma) := \frac{1}{Z_{\beta,\omega}} \exp\left[\beta H_\omega(\sigma)\right].$$

For many considerations, it is important to investigate properties of a replicated system, by taking, for fixed ω, product measures of $\mathcal{G}_\omega$:

$$\mathcal{G}_\omega^{\otimes n}(\sigma) := \prod_{i=1}^n \mathcal{G}_\omega\left(\sigma^i\right), \ \sigma = \left(\sigma^1, \ldots, \sigma^n\right)$$

on Σ^n. The $\sigma^1, \ldots, \sigma^n$ are called "replicas". Typically, we write $\mathcal{E}^{(n)}$ or sometimes simply $\langle \cdot \rangle$ for the expectations under this measure: If $\Phi : \Sigma^n \to \mathbb{R}$

$$\mathcal{E}^{(n)}\Phi := \sum_\sigma \Phi(\sigma) \mathcal{G}_\omega^{\otimes n}(\sigma).$$

It is important to note that this expectation is still a random variable defined on $(\Omega, \mathcal{F}, \mathbb{P})$.

Of crucial importance for many investigations are the measures $\nu^{(n)}$ on $\Omega \times \Sigma^n$ given by

$$\nu^{(n)}(d\omega, \sigma) := \mathbb{P}(d\omega)\, \mathcal{G}_\omega^{\otimes n}(\sigma),$$

or it's marginal on Σ^n

$$\nu^{(n)}(\sigma) := \int_\Omega \mathcal{G}_\omega^{\otimes n}(\sigma)\, \mathbb{P}(d\omega). \tag{1.1}$$

We don't distinguish in notations between the two, as it will always be clear from the context if we consider the measure on $\Omega \times \Sigma^n$ or only its marginal on Σ^n. It is important to note that $\nu^{(n)}$ is *not* the product measure of $\nu^{(1)}$, but it is important to note that the marginal of $\nu^{(n+1)}$ on Σ^n is $\nu^{(n)}$. We therefore often drop the index n in $\nu^{(n)}$, and write just ν, if there it is clear from the context how many replicas are considered.

Often, we write expectations of a function $\Phi : \Sigma^n \to \mathbb{R}$ with respect to $\nu^{(n)}$ just as $\nu^{(n)}(\Phi)$ or $\nu(\Phi)$.

1.3 Gaussian random variables

Let $N \in \mathbb{N}$, and γ_N be the standard normal distribution on $\mathbb{R}^N$. γ_N has the density $(2\pi)^{-N/2} \exp\left[-|\mathbf{x}|^2/2\right]$ with respect to Lebesgue measure, where $|\mathbf{x}|$ denotes the Euclidean norm. γ_N is invariant under rotations, i.e. if $\phi : \mathbb{R}^N \to \mathbb{R}^N$ is an orthogonal map, then $\gamma_N \phi^{-1} = \gamma_N$. If $f : \mathbb{R}^N \to \mathbb{R}$ is Lipshitz continuous then it is integrable with respect to γ_N. We write $\gamma_N(f)$ for its expectation.

Theorem 1.1
Let $f : \mathbb{R}^N \to \mathbb{R}$ be Lipshitz continuous with

$$\|f\|_{\mathrm{Lip}} := \sup_{x \neq y} \frac{|f(x) - f(y)|}{|x - y|} < \infty.$$

Then for any $t > 0$

$$\gamma_N\left(\left\{x \in \mathbb{R}^N : |f(x) - \gamma_N(f)| > t\right\}\right) \leq 2 \exp\left[-t^2/2\, \|f\|_{\mathrm{Lip}}^2\right].$$

Proof. See e.g. [32] ■

The second result we need from Gaussian variables is Wick's identity:

Theorem 1.2
Let $(X_1, \ldots, X_d)$ be a centered Gaussian random vector with covariance matrix $\Gamma = (\gamma_{ij})$, and let $\Phi : \mathbb{R}^d \to \mathbb{R}$ be a continuously differentiable function with partial derivatives $\partial_j \Phi$, satisfying for some $C > 0$

$$|\Phi(x_1, \ldots, x_d)| \leq C \exp[C|\mathbf{x}|],$$

where $|\mathbf{x}|$ *is the Euclidean norm of* $\mathbf{x} = (x_1, \ldots, x_d)$. *Then*

$$E\left(X_i \Phi\left(X_1, \ldots, X_d\right)\right) = \sum_j \gamma_{ij} E \partial_j \Phi\left(X_1, \ldots, X_d\right).$$

Proof. For the case of i.i.d. standard Gaussians X_i, the statement is

$$E\left(X_i \Phi\left(X_1, \ldots, X_d\right)\right) = E \partial_i \Phi\left(X_1, \ldots, X_d\right),$$

and it suffices to consider the case $d = 1$. For that special case, it is partial integration

$$\begin{aligned} E\left(X\Phi\left(X\right)\right) &= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} \mathrm{e}^{-x^2/2} x \Phi\left(x\right) dx \\ &= -\frac{1}{\sqrt{2\pi}} \mathrm{e}^{-x^2/2} \Phi\left(x\right) \Big|_{x=-\infty}^{\infty} + \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} \mathrm{e}^{-x^2/2} \Phi'\left(x\right) dx, \end{aligned}$$

and the first term vanishes by the growth condition on Φ, and so the rhs is $E\Phi'\left(X\right)$.

For the general case, we represent the X's through a linear transformation of i.i.d. Gaussians ξ_i :

$$X_i = \sum_{j=1}^{d} a_{ij} \xi_j,$$

where the matrix $A = (a_{ij})$ satisfies $AA^T = \Gamma$. Then

$$E\left(X_i \Phi\left(X_1, \ldots, X_d\right)\right) = \sum_j a_{ij} E \xi_j \Phi\left(A\xi\right),$$

ξ as a column vector, and

$$\begin{aligned} E\xi_j \Phi\left(A\xi\right) &= \sum_l a_{lj} E \partial_l \Phi\left(A\xi\right) \\ &= \sum_l a_{lj} E \partial_l \Phi\left(X\right). \end{aligned}$$

This proves the claim. ■

We give a simple application which will be used several times.

Let Σ be a finite set, and $\{H\left(\sigma\right)\}_{\sigma \in \Sigma}$ be a Gaussian vector. Define the Gibbs distribution and $\nu^{(n)}$ as in Section 1.2.

Consider an additional centered Gaussian random vector $\{X\left(\sigma\right)\}_{\sigma \in \Sigma}$ such that $\{X\left(\sigma\right), H\left(\sigma\right)\}_{\sigma \in \Sigma}$ is jointly Gaussian.

Proposition 1.3
Under the above conditions

$$\begin{aligned} \nu^{(n)}\left[X\left(\sigma^1\right)\Phi\left(\sigma\right)\right] &= \nu^{(n)}\left[\Phi\left(\sigma\right)\sum\nolimits_{i=1}^{n} \operatorname{cov}\left(X\left(\sigma^1\right), H\left(\sigma^i\right)\right)\right] \\ &\quad - n\nu^{(n)}\left[\Phi\left(\sigma\right)\operatorname{cov}\left(X\left(\sigma^1\right), H\left(\sigma^{n+1}\right)\right)\right]. \end{aligned}$$

with $\sigma = \left(\sigma^1, \ldots, \sigma^n\right)$. *(In the second summand, the* $(n+1)$*-th replica* σ^{n+1} *enters only via* H*).*

Proof.

$$\nu^{(n)}\left[X\left(\sigma^1\right)\Phi\left(\sigma\right)\right]=\sum_{\sigma}\Phi\left(\sigma\right)\mathbb{E}X\left(\sigma^1\right)\frac{1}{Z^n}\exp\left[\sum\nolimits_{i=1}^{n}H\left(\sigma^i\right)\right].$$

From Theorem 1.2, one obtains

$$\mathbb{E}X\left(\sigma^1\right)\frac{1}{Z^n}\exp\left[\sum\nolimits_{i=1}^{n}H\left(\sigma^i\right)\right]=\sum_{i=1}^{n}\operatorname{cov}\left(X\left(\sigma^1\right),H\left(\sigma^i\right)\right)\mathbb{E}\frac{1}{Z^n}\exp\left[\sum\nolimits_{i=1}^{n}H\left(\sigma^i\right)\right]$$
$$-n\sum_{\sigma\in\Sigma}\mathbb{E}\frac{1}{Z^{n+1}}\operatorname{cov}\left(X\left(\sigma^1\right),H\left(\sigma\right)\right)\exp\left[\sum\nolimits_{i=1}^{n}H\left(\sigma^i\right)+H\left(\sigma\right)\right].$$

We may denote the new summation over σ as a summation over a "new" variable σ^{n+1}. Implementing this, we get the claimed expression. ■

Remark 1.4
The H-variables don't have to be centered. Writing them as $H\left(\sigma\right)=H_0\left(\sigma\right)+a\left(\sigma\right)$, $a\left(\sigma\right)\in\mathbb{R}$, and H_0 centered, one evidently gets the same formula.

1.4 Point processes

The point processes we consider are all either on $\mathbb{R}$, $\mathbb{R}^+$ or $\mathbb{R}^d$. We write X for either of these spaces, being more specific when needed. We just give a summary of the fact which are relevant for us. For proofs, see for instance [31] or [21].

The Borel-σ-field on X is denoted by $\mathcal{X}$. A measure μ on $(X,\mathcal{X})$ is called Radon measure if $\mu\left(K\right)<\infty$ for any compact $K\subset X$. We write RAD_X for the set of Radon measures on $(X,\mathcal{X})$. We can equip RAD_X with the topology of vague convergence which is generated by the evaluation mappings $\mu\longmapsto\int f d\mu$, $f\in C_0\left(X\right)$, where $C_0\left(X\right)$ denotes the set of continuous functions $X\to\mathbb{R}$ of compact support. We leave out the index X if there is no danger of confusion. It is known that on RAD there exists a metric ρ which is complete, and such that RAD has a countable dense subset, which is a metric for vague convergence. The Borel σ-field is denoted by $\mathcal{B}_{\mathrm{RAD}}$.

A sequence $\{Q_n\}_{n\in\mathbb{N}}$ of probability measures on $(\mathrm{RAD},\mathcal{B}_{\mathrm{RAD}})$ is said to converge weakly to a probability measure Q if

$$\lim_{n\to\infty}\int F\left(\mu\right)Q_n\left(d\mu\right)=\int F\left(\mu\right)Q\left(d\mu\right)$$

for any bounded continuous function $F:\mathrm{RAD}\to\mathbb{R}$.

A convenient tool is the Laplace functional. Let $\phi\in C_0^+\left(X\right)$. These are the non-negative functions in $C_0\left(X\right)$. If Q is a probability measure on $(\mathrm{RAD},\mathcal{B}_{\mathrm{RAD}})$, the Laplace functional L_Q on $C_0^+\left(X\right)$ is defined by

$$L_Q\left(\phi\right):=\int\exp\left[-\int\phi\ d\mu\right]Q\left(d\mu\right).$$

Proposition 1.5

a) *If* $L_Q(\phi) = L_{Q'}(\phi)$ *for all* ϕ, *then* $Q = Q'$.

b) *If* $\{Q_n\}$ *is a sequence of probability measures on* $(\mathrm{RAD}, \mathcal{B}_{\mathrm{RAD}})$, *and* Q *is a probability measure, then* $\{Q_n\}$ *converges weakly to* Q *if and only if*

$$\lim_{n\to\infty} L_{Q_n}(\phi) = L_Q(\phi)$$

holds for all $\phi \in C_0^+(X)$.

For proofs, see for instance [21] Prop. 11.1.VIII.
Of interest for us are only point measures on X, i.e. measures of the form

$$\sum_{i\in I} \delta_{x_i},$$

where $\{x_i\}$ is a finite or countable sequence in X which has the property that $\sum_i 1_K(x_i) < \infty$ for any compact subset $K \subset X$. The set of Radon measures of this form is denoted by PT_X. It is easy to see that this is a Borel subset of RAD_X.

Definition 1.6
A random variable Ξ *defined on some probability space* $(\Omega, \mathcal{F}, P)$ *with values in* $(\mathrm{PT}, \mathcal{B}_{\mathrm{PT}})$ *is called a* ***point process****.*

One can always realize such a point process through a finite or infinite sequence $\{\xi_k\}$ or X-valued random variables: $\Xi = \sum_k \delta_{\xi_k}$. The ordering of the random variables is irrelevant for the point process. If $X = \mathbb{R}$ or $\mathbb{R}^+$, and Ξ is almost surely a single point measure where the points have a largest element, then one can choose a fixed ordering of the points by ordering them downwards $\xi_1 > \xi_2 > \dots$. This is sometimes convenient.

Definition 1.7
Let μ *be a Radon measure on* $(X, \mathcal{X})$. *A point process* Ξ *is called a* ***Poisson point process with intensity measure*** μ *(supposed to be Radon) if the following two conditions are satisfied*

- *If* $A \subset X$ *has compact closure then* $\Xi(A)$ *is Poisson distributed with parameter* $\mu(A)$.
- *If* $A_1, \dots, A_n$ *are pairwise disjoint sets, then* $\Xi(A_1), \dots, \Xi(A_n)$ *are independent random variables.*

We say that Ξ is a $\mathrm{PPP}(\mu)$ if it is a Poisson point process with intensity measure μ.
For a point process, we write L_Ξ for the Laplace functional of its distribution:

$$L_\Xi(\phi) = \int \exp\left[-\int \phi \, d\mu\right] P\Xi^{-1}(d\mu) = E \exp\left[-\int \phi \, d\Xi\right].$$

Proposition 1.8
If Ξ is a PPP (μ) *then for all* $\phi \in C_0^+(X)$

$$L_\Xi(\phi) = \exp\left[-\int \left(1 - \mathrm{e}^{-\phi(x)}\right) \mu(dx)\right].$$

Proof. Let $\phi \in C_0^+(X)$. Given $\varepsilon > 0$, we can find finitely many $A_1, \ldots, A_n \in \mathcal{X}$ with compact closure, and nonnegative numbers $a_1, \ldots, a_n$ such that

$$\left\|\phi - \sum\nolimits_i a_i 1_{A_i}\right\|_\infty \leq \varepsilon.$$

Replacing ϕ by the simple function $s = \sum_i a_i 1_{A_i}$, we get

$$\begin{aligned} E\exp\left[-\int s d\Xi\right] &= E\exp\left[-\sum\nolimits_i a_i \Xi(A_i)\right] \\ &= \prod\nolimits_{i=1}^n E\exp\left[-a_i \Xi(A_i)\right], \end{aligned}$$

as the $\Xi(A_i)$ are independent. As they are Poisson with parameter $\mu(A_i)$ we get

$$\begin{aligned} E\exp\left[-a_i \Xi(A_i)\right] &= \mathrm{e}^{-\mu(A_i)} \sum_{k=0}^{\infty} \frac{\mu(A_i)^k}{k!} \mathrm{e}^{-k a_i} \\ &= \exp\left[-\mu(A_i)\left(\mathrm{e}^{-a_i} - 1\right)\right], \end{aligned}$$

i.e.

$$\begin{aligned} E\exp\left[-\int s d\Xi\right] &= \exp\left[-\sum\nolimits_i \mu(A_i)\left(\mathrm{e}^{-a_i} - 1\right)\right] \\ &= \exp\left[-\int \left(1 - \mathrm{e}^{-s(x)}\right) \mu(dx)\right]. \end{aligned}$$

The result now follows by an approximation procedure. ■

A basic result in point process theory is:

Theorem 1.9
For any Radon measure μ on $(X, \mathcal{X})$, a PPP (μ) *exists.*

For a proof, see e.g. [21].

We are interested only in the case where X is $\mathbb{R}$ or $\mathbb{R}^d$ or an open subset of these spaces, and where μ has a density with respect to Lebesgue measure. If g is such a density, we say that a PPP (μ) is a Poisson point process with density g, and sometimes write PPP $(g(t)\, dt)$.

If X, X' are two separable, locally compact metric spaces, and $f : X \to X'$ is a continuous mapping, then f defines a mapping from measures μ on X to measures μf^{-1} on X'. However, if μ is Radon, then not necessarily, μf^{-1} is Radon. We therefore assume that f has the property that $f^{-1}(K)$ is compact in X whenever $K \subset X'$ is compact.

Lemma 1.10
Let $f : X \to X'$ be a continuous mapping such that $f^{-1}(K)$ is compact whenever $K \subset X'$ is compact. If Ξ is a PPP (μ), *then Ξf^{-1} is a* PPP $\left(\mu f^{-1}\right)$.

Proof. Check the Laplace functional. ■

2 Examples of spin glasses

The most "natural" example of a spin glass is the

2.1 Edwards-Anderson model

This is a spin glass version of the standard Ising model. Take $\Lambda_N := \{1, \ldots, N\}^d$, $\Sigma_N := \{-1, 1\}^{\Lambda_N}$. For $\sigma = \{\sigma_x\}_{x \in \Lambda_N} \in \Sigma_N$, $\sigma_x \in \{-1, 1\}^N$, take

$$H_{N,\omega}(\sigma) := \sum_{i \sim j \in \Lambda_N} g_{ij}(\omega)\, \sigma_i \sigma_j,$$

where g_{ij} are i.i.d. standard Gaussian random variables, defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and the sum is over unordered nearest neighbor pairs. (g_{ij} should depend only on $\{i, j\}$, not on the ordering). The partition function is defined by

$$Z_{N,\beta,\omega} := \sum_{\sigma} \exp\left[\beta H_{N,\omega}(\sigma)\right],$$

and the Gibbs measure

$$\mathcal{G}_{N,\beta,\omega}(\sigma) := \frac{1}{Z_{N,\beta,\omega}} \exp\left[\beta H_{N,\omega}(\sigma)\right].$$

Natural questions are to determine the **free energy:**

$$f(\beta) := \lim_{N \to \infty} \frac{1}{N} \log Z_{N,\beta,\omega}.$$

The following result is not difficult to prove, using the subadditive ergodic theorem, and Gaussian concentration:

Proposition 2.1
$f(\beta)$ exists and

$$f(\beta) = \lim_{N \to \infty} \frac{1}{N} \mathbb{E} \log Z_{N,\beta}.$$

In particular, the free energy is non-random.

I leave the proof as an exercise. It will play no role in what follows.
The fact that

$$\lim_{N \to \infty} \frac{1}{N} \log Z_{N,\beta,\omega} = \lim_{N \to \infty} \frac{1}{N} \mathbb{E} \log Z_{N,\beta}$$

is usually called **self-averaging** of the free energy.

For $d = 1$, the model can be discussed through a transfer matrix approach using the theory of products of random matrices.

Unfortunately, for $d \geq 2$, besides of elementary properties, essentially nothing is known about the EA-model. At the moment, there are no techniques available to evaluate or seriously discuss $f(\beta)$. There are some results for high temperature, i.e. small β, but for instance, the question about a phase transition and its properties is largely unknown.

The problems are stemming to a large extent from the fact that the model exhibits so-called frustration, because the g can take both positive and negative values. For instance, in $d = 2$, there will (for large N) be points $(i_1, i_2) \in \Sigma_N$ with $g_{(i_1,i_2),(i_1+1,i_2)} > 0$, $g_{(i_1,i_2),(i_1,i_2+1)} < 0$, $g_{(i_1+1,i_2),(i_1+1,i_2+1)} > 0$, $g_{(i_1,i_2+1),(i_1+1,i_2+1)} > 0$, which implies that under the Gibbs measure, $\sigma_{(i_1,i_2)}$ would "like" to be the same as $\sigma_{(i_1+1,i_2)}$, and this the same as $\sigma_{(i_1+1,i_2+1)}$, and the latter, because of $g_{(i_1,i_2+1),(i_1+1,i_2+1)} > 0$ to be the same as $\sigma_{(i_1,i_2+1)}$. However, $\sigma_{(i_1,i_2+1)}$ would like to have opposite sign to $\sigma_{(i_1,i_2)}$ because of $g_{(i_1,i_2),(i_1,i_2+1)} < 0$. This causes already a problem to find the ground state, i.e. the configuration σ which minimizes $H(\sigma)$ which would be a trivial problem if all g's would be positive. In fact, to find $\min_\sigma H(\sigma)$ is an enormously hard and unsolved problem for the EA model and this is then also reflected for the finite but large β Gibbs measure. For the high-temperature case, see [24].

For a recent survey about the problems arising in short range spin glasses, and also a discussion of possible relation with mean-field glasses, see [36].

One should add that the problems appearing here are very standard problems in classical probability theory. Remark that the collection $\{H_N(\sigma)\}_{\sigma\in\Sigma_N}$ is a family of centered Gaussian random variables. The covariance structure is easily computed:

$$\begin{aligned}\operatorname{cov}(H_N(\sigma), H_N(\tau)) &= \sum_{i\sim j\in\Lambda_N,\, i'\sim,j'\in\Lambda_N} \mathbb{E} g_{ij} g_{i'j'} \sigma_i\sigma_j\tau_{i'}\tau_{j'} \\ &= \sum_{i\sim j\in\Lambda_N,} \sigma_i\sigma_j\tau_i\tau_j.\end{aligned}$$

To find $\min_\sigma H_N(\sigma)$ therefore amount to find the minimum of Gaussian field of random variables which have a "simple" covariance structure.

2.2 Sherrington-Kirkpatrick model: A mean field version of the EA model

In a mean-field model, all the sites in the base space interact with any other on equal footing. There is then no point to assume a geometric structure of Λ_N and one just takes $\Lambda_N = \{1, \ldots, N\}$ and $\Sigma_N := \{-1, 1\}^N$. The Hamiltonian is defined by

$$H_{N,\omega}(\sigma) := \sum_{1\leq i<j\leq N} \frac{g_{ij}(\omega)}{\sqrt{N}} \sigma_i\sigma_j.$$

The g_{ij} are again standard Gaussian random variables. It is convenient to define the g_{ij} for all pairs of indices by putting $g_{ij} := g_{ji}$ for $i > j$, and $g_{ii} = 0$.

At first sight, when one compares with the Curie-Weiss model (see Section 9), the $\sqrt{N}$ normalization looks strange, but a moment's reflection reveals that it is the right one: A specific spin σ_i interacts with the other ones through

$$\frac{1}{\sqrt{N}} \sum_{j:j\neq i} g_{ij}\sigma_j.$$

This quantity, for fixed $\sigma_j,\ j \neq i$, is a Gaussian with variance $(N-1)/N$, so the influence of the other spins on σ_i is of order 1 with the above normalization. In this respect, the situation is the same as in the Curie-Weiss or the EA model.

Typically, one includes also a non-random external field with strength h, i.e. the Hamiltonian is

$$H_{N,h,\omega}(\sigma) := \sum_{1\leq i<j\leq N} \frac{g_{ij}(\omega)}{\sqrt{N}}\sigma_i\sigma_j + h\sum_i \sigma_i, \tag{2.1}$$

sometimes also random one:

$$H_{N,h,\omega}(\sigma) := \sum_{1\leq i<j\leq N} \frac{g_{ij}(\omega)}{\sqrt{N}}\sigma_i\sigma_j + \sum_i g_i(\omega)\,\sigma_i,$$

with new independent Gaussians g_i, but we stay mainly with (2.1). The partition function, and the Gibbs measure are then defined by

$$Z_{N,\beta,h,\omega} := \sum_\sigma \exp\left[\beta H_{N,h,\omega}(\sigma)\right],$$
$$\mathcal{G}_{N,\beta,h,\omega}(\sigma) := \frac{1}{Z_{N,\beta,h,\omega}} \exp\left[\beta H_{N,h,\omega}(\sigma)\right].$$

In the case of a non-vanishing external field, also h would get multiplied by β. That's the way physicists like it, but mathematically, there is no point to multiply h by β, and we set

$$Z_{N,\beta,h,\omega} := \sum_\sigma \exp\left[\beta \sum_{1\leq i<j\leq N} \frac{g_{ij}(\omega)}{\sqrt{N}}\sigma_i\sigma_j + h\sum_i \sigma_i\right],$$

and the Gibbs measure accordingly.

The covariance structure of the Hamiltonian is easily computed:

$$\begin{aligned}\mathbb{E}\left(\sum_{1\leq i<j\leq N} \frac{g_{ij}}{\sqrt{N}}\sigma_i\sigma_j \sum_{1\leq i<j\leq N} \frac{g_{ij}}{\sqrt{N}}\tau_i\tau_j\right) &= \frac{1}{N}\sum_{1\leq i<j\leq N} \sigma_i\sigma_j\tau_i\tau_j \\ &= \frac{1}{2N}\sum_{i,j=1}^N \sigma_i\sigma_j\tau_i\tau_j - \frac{1}{2} \\ &= \frac{N}{2}R_N(\sigma,\tau)^2 - \frac{1}{2},\end{aligned}$$

where the **overlap** is defined by the inner product:

$$R_N(\sigma,\tau) = \frac{1}{N}\sum_{i=1}^{N}\sigma_i\tau_i. \tag{2.2}$$

2.3 The perceptron

The perceptron is a particular neural net. In its simplest form, one has M patterns $\left(S_i^k\right)_{1\le i\le N}$, $1 \le k \le M$, of ± 1 and one has to find "neural net parameters" σ_i which produce 1 as the output of sign $\left(\sum_i \sigma_i S_i^k\right)$, $k = 1,\ldots,M$. There are many more versions, for instance where the function sign (x) is replaced by another one, and mainly, where the network has several "layers", i.e. where the output of a first layer is the input for a second layer, and so on. Also there can be (and typically is in networks which are applied in practice) a complicated pattern how the outputs are used in the next layer. We completely neglect these subtleties and ask only how many patterns in the single layer perceptron can be stored safely, meaning how big can M be such that σ's are found which give output one to all the patterns.

That's not really a well posed problem as the answer will depend very much on the interrelations between the patterns. To simplify further, we assume that the patterns are randomly chosen. Further simplifications arise by assuming that the S_i^k are i.i.d. Gaussians: write g_{ik} instead, and the σ_i are ± 1. These latter simplifications are actually not so important, and with some efforts, many of the results which were obtained (rigorously mainly by Talagrand, or non-rigorously mainly by Gardner and Derrida) can be generalized relaxing the conditions, but not the basic independence of the patterns. In the above framework, the problems have some similarity with the problems in the SK model. Let's first define

$$H_k := \left\{x \in \mathbb{R}^N : \sum\nolimits_i x_i g_{ik} \ge 0\right\},\ k = 1,\ldots,M$$

which are random half spaces defined by the patterns. Then question about the existence of neural net parameters σ is if $\bigcap_{k=1}^M H_k \cap \Sigma_N$ is "typically" non-empty where $\Sigma_N := \{-1,1\}^N$.

Bernard Derrida and Elizabeth Gardner in the late eighties derived by non-rigorous replica computations a number remarkable results. Finally, Talagrand gave rigorous proofs for a number of their results by a complicated version of his "cavity method" (Chap 2, 8, 9 of [42]).

A trivial observation is:

$$\begin{aligned}\mathbb{E}\left|\bigcap\nolimits_{k=1}^M H_k \cap \Sigma_N\right| &= \sum_{\sigma\in\Sigma_N}\mathbb{P}\left(\sigma \in \bigcap\nolimits_{k=1}^M H_k\right)\\ &= 2^N\mathbb{P}\left(\sigma \in \bigcap\nolimits_{k=1}^M H_k\right)\\ &= 2^N 2^{-M} = 2^{N-M}.\end{aligned}$$

Therefore, for $\alpha > 1$

$$\mathbb{P}\left(\bigcap_{k=1}^{\alpha N} H_k \cap \Sigma_N \neq \emptyset\right) \to 0. \tag{2.3}$$

On the other hand, for $\alpha < 1$, $M = \alpha N$, the above expectation is exponentially growing in N. Does that mean that $\mathbb{P}\left(\bigcap_{k=1}^{\alpha N} H_k \cap \Sigma_N \neq \emptyset\right) \to 1$? The answer is of course "No", as otherwise there would be no point to publish on the problem. Here is one of Talagrand's result:

Theorem 2.2

a) There exists $\alpha < 1$ such that (2.3) holds.

b) For small $\alpha \in (0,1)$

$$\frac{1}{N}\log\left|\bigcap_{k=1}^{\alpha N} H_k \cap \Sigma_N\right| \to \log 2 + \mathrm{RS}(\alpha), \text{ a.s.}$$

$$\mathrm{RS}(\alpha) := -\frac{r}{2}(1-q) + E\log\cosh\left(\sqrt{q}Z\right) + \alpha E\log\phi\left(\left[\frac{-Z\sqrt{q}}{\sqrt{1-q}}, \infty\right)\right), \ Z \text{ Gaussian.}$$

where ϕ is the standard normal distribution, and r, q satisfy

$$q = E\tanh^2\left(\sqrt{r}Z\right), \ r = \frac{\alpha}{1-q}EF\left(\frac{-Z\sqrt{q}}{\sqrt{1-q}}\right), \ F(x) := \frac{1}{\sqrt{2\pi}}\frac{\exp\left[-x^2/2\right]}{P(Z \geq x)},$$

(a formula of which Talagrand wrote that "you should rush to require medical attention if it seems transparent to you").

There is a "soft" version of the problem. Given a smooth function $u : \mathbb{R} \to \mathbb{R}$, and the so-called "cavity variables",

$$y_{\sigma,k} := \frac{1}{\sqrt{N}}\sum_{i=1}^{N} g_{ik}\sigma_i \tag{2.4}$$

define

$$Z_{N,u,\alpha} := \sum_{\sigma} \exp\left[\sum\nolimits_{k=1}^{\alpha N} u\left(y_{\sigma,k}\right)\right]. \tag{2.5}$$

Our original problem corresponds to the special choice

$$u(x) = -\infty 1_{(-\infty,0)}(x), \text{ with } 0 \cdot \infty = 0.$$

With this choice, $\exp\left[\sum_{k=1}^{\alpha N} u\left(y_{\sigma,k}\right)\right] = 1$ if and only if $y_{\sigma,k} \geq 0$ for all k, and so

$$Z_{N,u,\alpha} = \left|\bigcap_{k=1}^{\alpha N} H_k \cap \Sigma_N\right|.$$

This choice for u is of course not "smooth" which creates a lot of problems. Talagrand first derived a formula for

$$f(u,\alpha) := \lim_{N\to\infty} \frac{1}{N} \log Z_{N,u,\alpha},$$

for smooth u (and small α), and finally, by a very complicated approximation procedure with a choice of N-dependent smooth u_N approximating $-\infty 1_{(-\infty,0)}$ he was able to prove the above theorem.

There are a number of reasons why the perceptron, i.e. the spin glass with Hamiltonian

$$H(\sigma) := \sum\nolimits_{k=1}^{\alpha N} u(y_{\sigma,k})$$

is interesting. For instance, the so-called bipartite SK-model with two sets of spin variables $\sigma_1,\ldots,\sigma_N,\tau_1,\ldots,\tau_N \in \{-1,1\}$ and Hamiltonian

$$H(\sigma,\tau) := \frac{1}{\sqrt{N}} \sum_{i=1}^{N}\sum_{j=1}^{N} g_{ij}\sigma_i\tau_k$$

can easily reduced to it: Summing out for instance the τ's, one gets a perceptron with $u(x) = \beta \log\cosh(x)$.

Despite the fact that the bipartite SK-model "looks" being simpler than the ordinary SK-model, much less is known about it (for large β).

There are other models which can be reduced to the perceptron, for instance the Hopfield model.

The form of the partition function (2.5) suggests a connection with classical large deviation theory: Defining with the "cavity" variables $y_{\sigma,k}$ from (2.4) the empirical distribution

$$L_{N,\alpha,\sigma}(\omega) := \frac{1}{\alpha N} \sum_{k=1}^{\alpha N} \delta_{y_{k,\sigma}(\omega)}$$

one can write

$$\sum_{k=1}^{[\alpha N]} u(y_{k,\sigma}(\omega)) := \alpha N \int u(x)\, L_{N,\alpha,\sigma}(dx),$$

i.e. the Hamiltonian is simply a linear function of the empirical measure. One may ask if there is a "quenched" large deviation principle for $L_{N,\alpha,\sigma}$. For a given fixed σ, this is of course nothing but the classical Sanov theorem as the $y_{k,\sigma}$ are independent in k, and standard normally distributed. Therefore, for fixed σ, one has

$$\limsup_{N\to\infty} \frac{1}{N} \log \mathbb{P}(L_{N,\alpha,\sigma} \in A) \le -\inf_{\mu\in A} I(\mu|\phi),\ A \text{ closed } \subset \mathcal{M}_1^+(\mathbb{R})$$
$$\limsup_{N\to\infty} \frac{1}{N} \log \mathbb{P}(L_{N,\alpha,\sigma} \in U) \ge -\inf_{\mu\in U} I(\mu|\phi),\ U \text{ open } \subset \mathcal{M}_1^+(\mathbb{R}),$$

where $I(\mu|\phi)$ is the usual relative entropy of μ w.r.t. the standard normal distribution ϕ, and where $\mathcal{M}_1^+(\mathbb{R})$ is the set of probability measures on $\mathbb{R}$.

The proper formulation for our problem is to look at

$$\mathcal{N}_N(A,\omega) := \#\{\sigma : L_{N,\alpha,\sigma,\omega} \in A\},$$

so that

$$\mathbb{E}\mathcal{N}_N(A) = 2^N \mathbb{P}(L_{N,\alpha,\sigma} \in A),$$

where the RHS is independent of σ. The proper question is

Conjecture 2.3
There exists a "rate function" $J : \mathcal{M}_1^+(\mathbb{R}) \to [0,\infty]$ such that

$$\limsup_{N\to\infty} \frac{1}{N} \log \mathcal{N}_N(A) \le \log 2 - \inf_{\mu\in A} J(\mu), \ A \text{ closed } \subset \mathcal{M}_1^+(\mathbb{R}), \text{ a.s.}$$
$$\liminf_{N\to\infty} \frac{1}{N} \log \mathcal{N}_N(A) \ge \log 2 - \inf_{\mu\in U} J(\mu), \ U \text{ open } \subset \mathcal{M}_1^+(\mathbb{R}), \text{ a.s..}$$

This is beyond reach at the moment. Such a result has been proved for a "perceptron version" of the generalized random energy model in [15].

2.4 Combinatorial optimization: The assignment problem

There are similar models of interest in combinatorics, for instance in combinatorial optimization. One such case is the optimal assignment problem. In the simplest case one has twice N objects, say N girls and N boys. For every girl i and boy j, there is a mutual "satisfaction" of matching i with j, say U_{ij}. The problem is to find a perfect matching, i.e. an assignment of girls to the boys such that the sum of the satisfactions is maximal. Mathematically formulated, one is looking at

$$S_N = \max_\pi \sum_{i=1}^N U_{i\pi(i)},$$

the maximum running over all permutations of N elements. We assume now that the U_{ij} are i.i.d. uniformly distributed on $[0,1]$. Mathematically, it is the same whether we are maximizing the satisfaction or minimizing it. The latter is formally slightly more convenient. Of course, we could try to find a matching such that for any i, $\pi(i)$ is chosen that $U_{i\pi(i)} = \min_j U_{ij}$, but a moments reflection shows that this will not work as there may be different girls i which would choose the same boy, something which is forbidden. It however turns out that $\sum_i \min_j U_{ij}$ is not so far off from S_N. A simple computation gives that

$$\mathbb{E}\min_j U_{ij} = \frac{1}{N} + o\left(\frac{1}{N}\right),$$

and therefore

$$\mathbb{E}\sum_i \min_j U_{ij} = 1 + o(1).$$

It is a mathematical proved result, that

$$\lim_{N\to\infty} \mathbb{E}S_N = \frac{\pi^2}{6}. \tag{2.6}$$

This was first derived in the physics literature by regarding it as spin glass problem. One introduces a finite temperature model by taking $\beta > 0$, and setting

$$F_{\beta,N} = \frac{1}{N} \log \sum_{\pi} \exp\left[-\beta N \sum\nolimits_i U_{i\pi(i)}\right],$$

one lets $N \to \infty$, and then divide it by β, and lets $\beta \to \infty$. The outcome from spin glass computation (see [34]) was that the limit is indeed $\pi^2/6$. However, this was by no means a mathematically rigorous proof. A proof of (2.6) was first given by David Aldous in 2001 [4]. From the spin glass theory viewpoint however, the problem is not very interesting and rather "trivial", as it does not exhibit the so-called "replica symmetry breaking" like the SK model. For a thorough discussion of the model from a spin glass viewpoint, see [33].

2.5 The simplest spin glass: The Random Energy Model (REM)

The main difficulty of the SK-model is coming from the fact that the Gaussian random variables (2.1) are correlated. Derrida [22] had the idea to ask if something interesting is happening if one just considers i.i.d. random variables as the Hamiltonian. However, one wants to keep the variance of the right order. The SK-Hamiltonian has a variance of order N. We assume that the variance is exactly N. Evidently, then also the σ need not to carry an internal structure. We therefore assume that we have just 2^N independent Gaussian random variables, call them $\omega \to H_{N,\omega}(\sigma)$, $1 \le \sigma \le 2^N$, defined on some probability space $(\Omega, \mathcal{F}, \mathbb{P})$, which are centered and have variance N. Of course, one may still assume that $\sigma \in \Sigma_N$, but this will be of no relevance here. We then define the "Gibbs measure" on the σ by defining for any $\omega \in \Omega$, and any $\beta > 0$

$$\mathcal{G}_{N,\beta,\omega}(\sigma) = \frac{\exp\left[\beta H_{N,\omega}(\sigma)\right]}{Z_{N,\beta,\omega}}, \tag{2.7}$$

where $Z_{N,\beta,\omega} = \sum_\sigma \exp\left[\beta H_{N,\omega}(\sigma)\right]$. We leave the index ω typically out. The free energy is as usual defined by

$$f(\beta) = \lim_{N\to\infty} \frac{1}{N} \log Z_{N,\beta}.$$

In principle, this could still be a random variable, but we will see in a moment, that the limit exists almost $\mathbb{P}$-almost surely, and does not depend on ω. In fact, we have the following result:

Theorem 2.4
$f(\beta)$ exists almost surely and is given by

$$f(\beta) = \begin{cases} \frac{\beta^2}{2} + \log 2 & \text{if } \beta \le \sqrt{2\log 2} \\ \sqrt{2\log 2}\,\beta & \text{if } \beta \ge \sqrt{2\log 2} \end{cases}.$$

Remark 2.5
The high temperature (small β) value is the annealed free energy

$$\lim_{N\to\infty}\frac{1}{N}\log\mathbb{E}Z_{N,\beta}.$$

If $\mathbb{E}Z_\beta^2 \le C\left(\mathbb{E}Z_\beta\right)^2$, with C not depending on N, then one gets in a standard way (see the discussion in Section 3.2)

$$f(\beta)=\lim_{N\to\infty}\frac{1}{N}\log\mathbb{E}Z_{N,\beta}.$$

One should however observe that this method rarely gives

$$\lim_{N\to\infty}\frac{1}{N}\log Z_{N,\beta}=\lim_{N\to\infty}\frac{1}{N}\log\mathbb{E}Z_{N,\beta}$$

in the correct range of β. Also here

$$\begin{aligned}\mathbb{E}Z_{N,\beta}^2 &= \sum_{\sigma,\sigma'}\mathbb{E}\exp\left[\beta\left(H_N(\sigma)+H_N\left(\sigma'\right)\right)\right]\\ &= \sum_{\sigma}\exp\left[2\beta^2N\right]+\sum_{\sigma\neq\sigma''}\exp\left[\beta^2N\right]\\ &= \exp\left[2\beta^2N+N\log 2\right]+2^N\left(2^N-1\right)\exp\left[N\beta^2\right].\end{aligned}$$

The first summand dominates the second as soon as $\beta>\sqrt{\log 2}$, and in fact, $\frac{\mathbb{E}Z_{N,\beta}^2}{\left(\mathbb{E}Z_{N,\beta}\right)^2}$ is exponentially growing in this case. One therefore sees that the second moment method fails to prove $f(\beta)=\beta^2/2+\log 2$ in the full region.

Proof of Theorem 2.4. The trick is to apply the "second moment method" not directly to Z but to

$$A_N(s)\overset{\text{def}}{=}\#\{\sigma : H_N(\sigma)\ge sN\}. \tag{2.8}$$

Let Φ as usual be the standard normal distribution function. Then

$$\mathbb{E}A_N(s)=2^N\left(1-\Phi\left(s\sqrt{N}\right)\right)\asymp 2^N\mathrm{e}^{-s^2N/2}.$$

Here we use the following notation: Given two sequences $\{a_N\},\{b_N\}$ of positive real numbers, which may depend on other parameters (like s above), then we write $a_N\asymp b_N$, provided for any $\varepsilon>0$ there exists N_0 (which may depend on the auxiliary parameters), such that

$$\mathrm{e}^{-\varepsilon N}a_N\le b_N\le\mathrm{e}^{\varepsilon N}a_N$$

for $N\ge N_0$.

For $s > \sqrt{2\log 2}$, $\mathbb{E}A_N(s)$ converges to 0, exponentially fast in N. From the Markov inequality, one gets $\mathbb{P}\left(A_N\left(s\right) \neq 0\right)$ converges to 0, exponentially fast, and then by Borel-Cantelli argument, we get that $A_N\left(s\right) = 0$ for large enough N, a.s. As this holds true for all $s > \sqrt{2\log 2}$, we get

$$\mathbb{P}\left(\limsup_{N\to\infty} \frac{1}{N} \sup_{\sigma} H_N\left(\sigma\right) \leq \sqrt{2\log 2}\right) = 1 \tag{2.9}$$

For the second moment, we obtain

$$\mathbb{E}A_N(s)^2 = 2^N\left(1 - \Phi\left(s\sqrt{N}\right)\right) + 2^N(2^N - 1)\left(1 - \Phi\left(s\sqrt{N}\right)\right)^2,$$

we see that for $0 \leq s < \sqrt{2\log 2}$, this is $\left[\mathbb{E}A_N(s)\right]^2$, up to a factor, which is exponentially close to 1. From that we get

$$A_N\left(s\right) \asymp \mathbb{E}A_N(s) = 2^N\left(1 - \Phi\left(s\sqrt{N}\right)\right) \asymp \exp\left[N\left(\log 2 - \frac{s^2}{2}\right)\right].$$

Summarizing, we get

$$A_N\left(s\right) \asymp \begin{cases} 0 & \text{if } s > \sqrt{2\log 2} \\ \exp\left[N\left(\log 2 - \frac{s^2}{2}\right)\right] & \text{if } s \in \left[0, \sqrt{2\log 2}\right) \\ 2^N & \text{if } s \leq 0 \end{cases}$$

with high probability. Using

$$\sum_{\sigma} \mathrm{e}^{\beta H_N(\sigma)} = N\beta \int_{-\infty}^{\infty} A_N(s)\mathrm{e}^{N\beta s} ds$$

the reader will have no difficulty to derive the theorem. ■

We next want to describe the large N behavior of the Gibbs measure $\mathcal{G}_{\omega,\beta,N}$. We have to distinguish between the high temperature case $\beta < \sqrt{2\log 2}$, and the low temperature case $\beta > \sqrt{2\log 2}$. We abstain from discussing the critical case $\beta = \sqrt{2\log 2}$. The fundamental difference is that in the high temperature case, the Gibbs measure is concentrated on a growing number of energy levels, which become dense and denser packed as $N \to \infty$. In contrast, in the low temperature regime, the Gibbs distribution is essentially concentrated on the top energy levels. We now make this precise.

Exercise 2.6
Assume $\beta < \sqrt{2\log 2}$

a) For any $\varepsilon > 0$, *there exist* $K, \delta > 0$ *such that*

$$\mathbb{P}\left(\left\{\omega : \mathcal{G}_{\omega,\beta,N}\left(\left\{\sigma : X_\sigma \in \left[\beta N - K\sqrt{N}, \beta N + K\sqrt{N}\right]\right\}\right) \geq 1 - \varepsilon\right\}\right) \geq 1 - \mathrm{e}^{-\delta N},$$

i.e. up to a negligible $\mathbb{P}$*-probability,* $\mathcal{G}$ *is concentrated* σ*'s for which the energy levels are in a window of size of order* $\sqrt{N}$ *around* βN. *(The fact that exactly* β *is the value where the energy levels concentrate under the Gibbs measure is an "accident".)*

b) $\max_\sigma P_{\omega,\beta,N}(\sigma)$ *is exponentially decaying,* $\mathbb{P}$*-a.s.*

The low temperature regime $\beta > \sqrt{2\log 2}$ is more interesting, as there, the energies get a macroscopic but random weight. For any sequence a_N of real numbers, $\sum_\sigma \delta_{X_\sigma - a_N}$ defines a point process on $\mathbb{R}$. We will sometimes just call such an object "the point process $\{X_\sigma - a_N\}_\sigma$".

Proposition 2.7
If $a_N = \sqrt{2\log 2}N - \frac{1}{2\sqrt{2\log 2}}\log N + \frac{1}{2\sqrt{2\log 2}}\log(2\pi)$, *then the above point process converges weakly to a* PPP $\left(\sqrt{2\log 2}\exp\left[-\sqrt{2\log 2}t\right]dt\right)$.

Proof. We denote by Q_N the law of $\sum_\sigma \delta_{X_a - a_N}$. If $\phi \in C_o^+(\mathbb{R})$, the

$$\begin{aligned} L_{Q_N}(\phi) &= \mathbb{E}\exp\left[-\sum_\sigma \phi\left(X_\sigma^{(N)} - a_N\right)\right] \\ &= \left\{\frac{1}{\sqrt{2\pi N}}\int \exp\left[-\phi(x-a_N) - \frac{x^2}{2N}\right]dx\right\}^{2^N} \\ &= \left\{1 - \frac{1}{\sqrt{2\pi N}}\int\left(1-\mathrm{e}^{-\phi(x)}\right)\exp\left[-\frac{(x+a_N)^2}{2N}\right]dx\right\}^{2^N}. \end{aligned}$$

We abbreviate

$$\delta(x,N) := \frac{1}{\sqrt{2\pi N}}\left(1-\mathrm{e}^{-\phi(x)}\right)\exp\left[-\frac{(x+a_N)^2}{2N}\right],$$

and so

$$L_{Q_N}(\phi) = \exp\left[2^N \log\left(1 - \int \delta(x,N)\,dx\right)\right].$$

As ϕ has compact support, there exist $K > 0$ such that $\phi = 0$ outside $[-K,K]$, and therefore $\delta(x,N) = 0$, too, outside this interval. On the other hand

$$\exp\left[-\frac{(x+a_N)^2}{2N}\right] = \sqrt{4\pi\log 2}\mathrm{e}^{-x\sqrt{2\log 2}}\exp\left[-N\log 2\right]\sqrt{N}(1+o(1)),$$

uniformly in $x \in [-K,K]$, and therefore

$$\delta(x,N) = 2^{-N}\left(1-\mathrm{e}^{-\phi(x)}\right)\sqrt{2\log 2}\mathrm{e}^{-x\sqrt{2\log 2}}(1+o(1)),$$

uniformly in $x \in [-K,K]$. Expanding $\log(1-\varepsilon) = -\varepsilon - O\left(\varepsilon^2\right)$ for ε small, it follows from the fact that $\delta(x,N) = 0$ outside $[-K,K]$:

$$\begin{aligned} &\exp\left[2^N \log\left(1 - \int \delta(x,N)\,dx\right)\right] \\ &= \exp\left[-\int\left(1-\mathrm{e}^{-\phi(x)}\right)\sqrt{2\log 2}\mathrm{e}^{-x\sqrt{2\log 2}}dx(1+o(1)) + O\left(2^{-N}\right)\right], \end{aligned}$$

i.e.

$$\lim_{N\to\infty} L_{Q_N}(\phi) = \exp\left[-\sqrt{2\log 2}\int\left(1-\mathrm{e}^{-\phi(x)}\right)\exp\left[-\sqrt{2\log 2}x\right]dx\right].$$

■

We next have to discuss the properties and some transformations of a PPP $(a\mathrm{e}^{-ax}dx)$ $\Xi = \{\xi_i\}$ on $\mathbb{R}$ where the parameter a is > 0. As $\int a\mathrm{e}^{-ax}dx = \infty$, the point process has infinitely many points. On the other hand, as the density is rapidly decaying for $x \to \infty$, it is evident that there are only finitely many points on the positive real axis. So, there is almost surely a largest point, and we can order the points downwards. For $\beta > 0$, the point process $\left\{\mathrm{e}^{\beta\xi_i}\right\}$ is again a Poisson point process on $\mathbb{R}^+$, which, according to Lemma 1.10 has as its intensity measure the one obtained from $\mu(dx) = a\mathrm{e}^{-ax}dx$ under the transformation f given by $f(x) = \mathrm{e}^{\beta x}$ with inverse $f^{-1}(x) = \beta^{-1}\log x$. So the resulting Poisson point process has density

$$a\exp\left[-a\beta^{-1}\log x\right]\frac{1}{\beta x} = a\beta^{-1}x^{-a\beta^{-1}-1}.$$

We set $\zeta := a\beta^{-1}$.

Lemma 2.8
Assume $0 < \zeta < 1$ and $\{\eta_i\}$ be a PPP $\left(\zeta x^{-\zeta-1}dx\right)$ on $\mathbb{R}^+$. Then the point process has infinitely many points, but only finitely many above any $\varepsilon > 0$. Furthermore

$$\sum_i \eta_i < \infty, \text{ a.s.}$$

Proof. That there are only finitely many points above $\varepsilon > 0$ follows from the integrability of the density on $[\varepsilon, \infty)$. But there are infinitely many points as the density is not integrable over $\mathbb{R}^+$. On the other hand

$$\mathbb{E}\left(\sum_i \eta_i 1_{\eta_i\le 1}\right) = \int_0^1 x\zeta x^{-\zeta-1}dx < \infty,$$

as we assumed $\zeta < 1$. So $\sum_i \eta_i 1_{\eta_i\le 1} < \infty$ almost surely. As there are only finitely points above 1, we conclude that $\sum_i \eta_i < \infty$.

(Remark that $\mathbb{E}\left(\sum_i \eta_i\right) = \int_0^\infty x\zeta x^{-\zeta-1}dx = \infty$, but that does of course not exclude $\sum_i \eta_i < \infty$). ■

Remark 2.9
If $\zeta \ge 1$, then $\sum_i \eta_i = \infty$ almost surely.

Given a PPP $\left(\zeta x^{-\zeta-1}dx\right)$ $\{\eta_i\}$ with $\zeta < 1$ we can normalize the points by putting

$$\bar{\eta}_i := \frac{\eta_i}{\sum_j \eta_j}.$$

Then of course $\sum_i \bar{\eta}_i = 1$. We may regard $\{\bar{\eta}_i\}$ as a random probability distribution on $\mathbb{N}$. For that, we have to attach the points to set of natural numbers. This is typically

done by ordering the points downwards $\bar{\eta}_1 > \bar{\eta}_2 > \cdots$ which can always be done. It should be kept in mind that this ordering is not encoded into the notion of a point process, and we do it only when it is necessary, or convenient.

Definition 2.10
The point process $\{\bar{\eta}_i\}$ *is called the Poisson-Dirichlet point process with parameter* $\zeta \in (0,1)$. *We denote it by* $\mathrm{PD}(\zeta)$.

We want to combine this result with Proposition 2.7 to obtain the limiting point process of the Gibbs weights $\{X_\sigma\}$ of the REM, for $\beta > \sqrt{2\log 2}$. First, we realize that

$$\mathcal{G}_{\beta,N}(\sigma) = \frac{\exp[\beta X_\sigma]}{\sum_\tau \exp[\beta X_\sigma]} = \frac{\exp[\beta X_\sigma - \beta a_N]}{\sum_\tau \exp[\beta X_\sigma - \beta a_N]}.$$

As $\{X_\sigma - a_N\}_\sigma$ converges to a PPP $\left(\sqrt{2\log 2}\mathrm{e}^{-\sqrt{2\log 2}x}dx\right)$, the following result is plausible:

Exercise 2.11
Prove that for $\beta > \sqrt{2\log 2}$, *the point processes* $\{\mathcal{G}_{\beta,N}(\sigma)\}_{1\le\sigma\le 2^N}$ *converge as* $N \to \infty$ *weakly to* $\mathrm{PD}\left(\frac{\sqrt{2\log 2}}{\beta}\right)$.

The proof is not completely trivial as the operation of normalizing to a random probability distribution is not continuous in the standard vague topology. One needs a truncation argument to prove the result.

The remarkable claim of the Parisi theory is that the Poisson-Dirichlet point process appears quite generally as the limit point process of the Gibbs weights of the so-called "pure states". It has however to be remarked that the notion of a "pure state" has not been made rigorous for most of the models.

3 First properties of the SK-model

In this chapter, we use the basic Hamiltonian of the SK model (2.1).

3.1 Basic properties of the free energy

An important property is the self-averaging of the free energy:

Theorem 3.1

$$\limsup_{N\to\infty} \left| \frac{1}{N}\mathbb{E}\log Z_{\beta,h,N} - \frac{1}{N}\log Z_{\beta,h,N} \right| = 0, \text{ a.s.}$$

Proof. This follows by Theorem 1.1 applied to the functions $\varphi : \mathbb{R}^{N(N-1)/2} \to \mathbb{R}$ given by

$$\varphi(x) = \log \sum_\sigma \exp\left[\frac{\beta}{\sqrt{N}} \sum_{1\le i<j\le N} x_{ij}\sigma_i\sigma_j + h\sum_{i=1}^N \sigma_i \right].$$

Clearly

$$\left|\sum_{1\leq i<j\leq N} x_{ij}\sigma_i\sigma_j - \sum_{1\leq i<j\leq N} y_{ij}\sigma_i\sigma_j\right| = \left|\sum_{1\leq i<j\leq N} (x_{ij}-y_{ij})\,\sigma_i\sigma_j\right|$$
$$\leq \sqrt{\frac{N(N-1)}{2}}\sqrt{\sum_{1\leq i<j\leq N}(x_{ij}-y_{ij})^2}$$
$$= \|x-y\|\sqrt{\frac{N(N-1)}{2}} \leq \frac{N}{\sqrt{2}}\|x-y\|$$

by the Cauchy-Schwarz inequality, where $\|\cdot\|$ denotes the Euclidean norm on $\mathbb{R}^{N(N-1)/2}$. Therefore, with

$$\psi(x,\sigma) \stackrel{\text{def}}{=} \frac{\beta}{\sqrt{N}}\sum_{1\leq i<j\leq N} x_{ij}\sigma_i\sigma_j + h\sum_{i=1}^N \sigma_i$$

we obtain

$$\exp[\psi(y,\sigma)]\exp\left[-\frac{\beta\sqrt{N}}{\sqrt{2}}\|x-y\|\right] \leq \exp[\psi(x,\sigma)]$$
$$\leq \exp[\psi(y,\sigma)]\exp\left[\frac{\beta\sqrt{N}}{\sqrt{2}}\|x-y\|\right],$$

and therefore

$$|\varphi(x)-\varphi(y)| \leq \frac{\beta\sqrt{N}}{\sqrt{2}}\|x-y\|,$$

i.e. φ is Lipshitz with

$$\|\varphi\|_{\text{Lip}} \leq \frac{\beta\sqrt{N}}{\sqrt{2}}.$$

From Theorem 1.1 we obtain

$$\mathbb{P}\left(\left|\frac{1}{N}\log Z_N - \frac{1}{N}\mathbb{E}\log Z_N\right| \geq N^{-1/4}\right) \leq 2\exp\left[-\frac{\sqrt{N}}{\beta^2}\right]. \tag{3.1}$$

As

$$\sum_N \exp\left[-\frac{\sqrt{N}}{\beta^2}\right] < \infty,$$

it follows by the Borel-Cantelli Lemma that with $\mathbb{P}$-probability one, the set of $N \in \mathbb{N}$ with

$$\left|\frac{1}{N}\log Z_N - \frac{1}{N}\mathbb{E}\log Z_N\right| \geq N^{-1/4}$$

is finite. ■

As a consequence, one sees that if $\lim_{N\to\infty} N^{-1}\log Z_N$ exists, it is non-random. For the existence of the limit, one only has to investigate the limit of the expectation. This was mathematically an open problem for quite some time, and was first proved by Guerra and Toninelli [30].

Theorem 3.2

$$f(\beta,h)=\lim_{N\to\infty}\frac{1}{N}\mathbb{E}\log Z_{\beta,h,N}\in\mathbb{R}$$

exists.

Proof. The proof is due to Guerra and Toninelli [30]. It uses one of the basic tools on which much of the recent progress in spin glass theory relies, namely a clever interpolation argument.

Let $N_1,N_2\in\mathbb{N}$, and $N:=N_1+N_2$. We choose independent standard Gaussians g_{ij},g'_{ij},g''_{ij}, and define for $t\in[0,1]$ the Hamiltonian $H_t(\sigma)$ which depends on the g,g',g'' :

$$\begin{aligned}H_t(\sigma):=\beta\sqrt{\frac{t}{N}}\sum_{1\le i<j\le N}g_{ij}\sigma_i\sigma_j+\beta\sqrt{\frac{1-t}{N_1}}\sum_{1\le i<j\le N_1}g'_{ij}\sigma_i\sigma_j \\ +\beta\sqrt{\frac{1-t}{N_2}}\sum_{N_1<i<j\le N}g''_{ij}\sigma_i\sigma_j+h\sum_{i=1}^{N}\sigma_i.\end{aligned}\tag{3.2}$$

(We incorporate β into the Hamiltonian H_t). We will need the derivative with respect to t :

$$\begin{aligned}\frac{dH_t(\sigma)}{dt}=\frac{\beta}{2}\Bigg\{\sqrt{\frac{1}{tN}}\sum_{1\le i<j\le N}g_{ij}\sigma_i\sigma_j \\ -\sqrt{\frac{1}{(1-t)N_1}}\sum_{1\le i<j\le N_1}g'_{ij}\sigma_i\sigma_j-\sqrt{\frac{1}{(1-t)N_2}}\sum_{N_1<i<j\le N}g''_{ij}\sigma_i\sigma_j\Bigg\}\end{aligned}\tag{3.3}$$

Then we define the partition function

$$Z(t):=\sum_{\sigma\in\Sigma_N}\exp H_t(\sigma),$$

and the Gibbs measure $\mathcal{G}_t(\sigma):=\exp\left[H_t(\sigma)\right]/Z(t)$, with expectation $\mathcal{E}_t$. (Please always remember that these are *quenched* expectations, i.e. they still depend on ω). Evidently, $Z(1)$ is the partition function Z_N we are looking after (depending on the random variables g), and $Z(0)$ is the product of two of our partition function $Z_{N_1}(g')Z_{N_2}(g'')$, where the important point is that the factors are independent, as they depend on independent random variables.

Differentiating with respect to t, we get

$$\begin{aligned}\frac{d}{dt}\frac{1}{N}\mathbb{E}\log Z(t) &= \frac{1}{N}\mathbb{E}\frac{1}{Z(t)}\frac{dZ(t)}{dt} \\ &= \frac{1}{N}\sum_{\sigma\in\Sigma_N}\mathbb{E}\frac{\exp\left[H_t(\sigma)\right]}{Z(t)}\frac{dH_t(\sigma)}{dt}.\end{aligned}$$

For the derivative we implement the expression (3.3) getting

$$\frac{d}{dt}\frac{1}{N}\mathbb{E}\log Z(t) = S_1 - S_2 - S_3,$$

where

$$\begin{aligned}S_1 &:= \frac{\beta}{2\sqrt{t}N^{3/2}}\sum_{\sigma\in\Sigma_N}\sum_{1\le i<j\le N}\sigma_i\sigma_j\mathbb{E}g_{ij}\frac{\exp\left[H_t(\sigma)\right]}{Z(t)} \\ &= \frac{\beta}{2\sqrt{t}N^{3/2}}\sum_{i<j}\nu_t\left(\sigma_i\sigma_j g_{ij}\right),\end{aligned}$$

where we use ν_t as introduced in (1.1), here with the interpolated Hamiltonian. S_2, S_3 are similarly defined terms from the second and third summand of (3.3).

We can apply Proposition 1.3 with $n = 1$, $F(\sigma) = \sigma_i\sigma_j$, $X(\sigma) = g_{ij}$ (this latter does not depend on σ). So, we get

$$\begin{aligned}\nu_t\left(\sigma_i\sigma_j g_{ij}\right) &= \beta\sqrt{\frac{t}{N}}\left[\nu_t\left(\left(\sigma_i\sigma_j\right)^2\right) - \nu_t\left(\sigma_i\sigma_j\sigma_i'\sigma_j'\right)\right] \\ &= \beta\sqrt{\frac{t}{N}}\left[1 - \nu_t\left(\sigma_i\sigma_j\sigma_i'\sigma_j'\right)\right],\end{aligned}$$

with a replicated set σ' of spin variables. Please remember that ν is applied to a possible arbitrary number of replicas, here two, as explained in Section 1.2. So

$$\begin{aligned}S_1 &= \frac{\beta^2}{2N^2}\sum_{1\le i<j\le N}\left[1 - \nu_t\left(\sigma_i\sigma_j\sigma_i'\sigma_j'\right)\right] \\ &= \frac{\beta^2}{4}\left(1 - \nu_t\left(R_N\left(\sigma,\sigma'\right)^2\right)\right),\end{aligned}$$

with the overlap $R_N(\sigma,\sigma')$ defined by (2.2).

By a similar computation, one gets

$$\begin{aligned}S_2 &= \frac{\beta^2 N_1}{4N}\left(1 - \nu_t\left(R_{N_1}\left(\sigma,\sigma'\right)^2\right)\right) \\ S_3 &= \frac{\beta^2 N_2}{4N}\left(1 - \nu_t\left(R_{N_1,N_1+N_2}\left(\sigma,\sigma'\right)^2\right)\right)\end{aligned}$$

with

$$R_{N_1}\left(\sigma,\sigma'\right):=\frac{1}{N_1}\sum_{i=1}^{N_1}\sigma_i\sigma_i',$$

$$R_{N_1,N_1+N_2}\left(\sigma,\sigma'\right):=\frac{1}{N_2}\sum_{i=N_1+1}^{N_1+N_2=N}\sigma_i\sigma_i'$$

so that

$$R_N=\frac{N_1}{N}R_{N_1}+\frac{N_2}{N}R_{N_1,N_2}. \tag{3.4}$$

Plugging that into the computation for S_2 and S_3, we get

$$\frac{d}{dt}\frac{1}{N}\mathbb{E}\log Z\left(t\right)=S_1-S_2-S_3=-\frac{\beta^2}{4}\nu_t\left(R_N^2-\frac{N_1}{N}R_{N_1}^2-\frac{N_2}{N}R_{N_1,N_2}^2\right).$$

From (3.4), one gets

$$R_N^2\leq\frac{N_1}{N}R_{N_1}^2+\frac{N_2}{N}R_{N_1,N_2}^2$$

and therefore

$$\frac{d}{dt}\frac{1}{N}\mathbb{E}\log Z\left(t\right)\geq 0.$$

From that we conclude

$$\frac{1}{N}\mathbb{E}\log Z\left(1\right)\geq\frac{1}{N}\mathbb{E}\log Z\left(0\right)$$

$$\mathbb{E}\frac{1}{N}\log Z_N\geq\frac{N_1}{N}\mathbb{E}\frac{1}{N_1}\log Z_{N_1}+\frac{N_2}{N}\mathbb{E}\frac{1}{N_2}\log Z_{N_2}.$$

This is a superadditivity property of the sequence of real numbers $\mathbb{E}\frac{1}{N}\log Z_N$. Therefore, it follows that

$$f\left(\beta,h\right)=\lim_{N\to\infty}\mathbb{E}\frac{1}{N}\log Z_N$$

exists, and equals

$$\sup_N\mathbb{E}\frac{1}{N}\log Z_N.$$

In order to prove the theorem, it only remains to show that this supremum is finite, but this follows from Jensen's inequality

$$\mathbb{E}\frac{1}{N}\log Z_N\leq\frac{1}{N}\log\mathbb{E}Z_N,$$

and the supremum of the latter is finite by the annealed computation we had done before. We in fact have the following annealed bound (see the Proposition below).

$$f\left(\beta,h\right)\leq\frac{\beta^2}{4}+\log\cosh\left(h\right)+\log 2.$$

■

Finally, two simple properties

Proposition 3.3

a) $f(\beta, h)$ *is a convex function of* $(\beta, h) \in \mathbb{R}^+ \times \mathbb{R}$.

b)

$$f(\beta, h) \leq f^{\text{ann}}(\beta, h) := \lim_{N\to\infty} \frac{1}{N} \log \mathbb{E} Z_N \ \forall \beta, h$$

Proof. b) follows from Jensen. a) follows in the usual way from the Hölder inequality: Let $\beta, \beta' > 0$, and $h, h' \in \mathbb{R}$, and $\lambda \in [0,1]$. Put $\beta(\lambda) := \lambda\beta + (1-\lambda)\beta'$, $h(\lambda) := \lambda h + (1-\lambda)h'$. Then

$$\begin{aligned}
Z_{\beta(\lambda),h(\lambda),N} &= \sum_\sigma \exp\left[\frac{\lambda\beta + (1-\lambda)\beta'}{\sqrt{N}} \sum_{i<j} g_{ij}\sigma_i\sigma_j + \left(\lambda h + (1-\lambda)h'\right) \sum_i \sigma_i\right] \\
&= \sum_\sigma \left\{\exp\left[\frac{\beta}{\sqrt{N}} \sum_{i<j} g_{ij}\sigma_i\sigma_j + h\sum_i \sigma_i\right]\right\}^\lambda \\
&\times \left\{\exp\left[\frac{\beta'}{\sqrt{N}} \sum_{i<j} g_{ij}\sigma_i\sigma_j + h'\sum_i \sigma_i\right]\right\}^{1-\lambda} \\
&\leq \left\{\sum_\sigma \exp\left[\frac{\beta}{\sqrt{N}} \sum_{i<j} g_{ij}\sigma_i\sigma_j + h\sum_i \sigma_i\right]\right\}^\lambda \\
&\times \left\{\sum_\sigma \exp\left[\frac{\beta'}{\sqrt{N}} \sum_{i<j} g_{ij}\sigma_i\sigma_j + h'\sum_i \sigma_i\right]\right\}^{1-\lambda}
\end{aligned}$$

by the Hölder inequality. Therefore

$$\frac{1}{N} \log Z_{\beta(\lambda),h(\lambda),N} \leq \lambda \frac{1}{N} \log Z_{\beta,h,N} + (1-\lambda) \frac{1}{N} \log Z_{\beta',h',N}.$$

Going the the $N \to \infty$ limit, we get

$$f(\beta(\lambda), h(\lambda)) \leq \lambda f(\beta, h) + (1-\lambda) f\left(\beta', h'\right).$$

■

Exercise 3.4

Replace the Gaussian variables g_{ij} *in the Hamiltonian by i.i.d. symmetric Bernoulli variables* ξ_{ij} *taking values* ± 1. *Write* $Z^{\text{Bernoulli}}_{\beta,h,N}$ *for the corresponding partition function. Prove that*

$$\lim_{N\to\infty} \frac{1}{N} \mathbb{E} \log Z^{\text{Bernoulli}}_{\beta,h,N} = f(\beta, h)$$

for all β, h, *where the expectation on the left hand side is with respect to the Bernoulli-variables* ξ_{ij}, *and the right hand side is the SK free energy.*

Hint: *Interpolate in a suitable way between the SK-Hamiltonian and the Bernoulli one, and try to control the derivative.*

3.2 High temperature, no external field

This was first discussed by Aizenman, Lebowitz and Ruelle, and Fröhlich and Zegarlinski ([1], [26]):

Theorem 3.5
Assume $h = 0$ and $\beta \leq 1$. Then

$$f(\beta, 0) = \frac{\beta^2}{4} + \log 2.$$

Proof. The so-called annealed free energy is easily computed:

$$\begin{aligned} f^{\text{ann}}(\beta, 0) &= \lim_{N\to\infty} \frac{1}{N} \log \mathbb{E} \sum_{\sigma} \exp\left[\frac{\beta}{\sqrt{N}} \sum_{1\leq i<j\leq N} g_{ij}\sigma_i\sigma_j\right] \\ &= \log 2 + \lim_{N\to\infty} \frac{1}{N} \log \exp\left[\frac{\beta^2}{2N} \sum_{1\leq i<j\leq N} \sigma_i^2\sigma_j^2\right] \\ &= \frac{\beta^2}{4} + \log 2. \end{aligned}$$

Therefore, the claim is that for $\beta \leq 1$, the free energy agrees with the annealed one. The proof is based on the "second moment method". We compute $\mathbb{E}Z^2$:

$$\begin{aligned} \mathbb{E}Z^2_{\beta,0,N} &= \sum_{\sigma,\tau} \mathbb{E} \exp\left[\frac{\beta}{\sqrt{N}} \sum_{1\leq i<j\leq N} g_{ij}\left(\sigma_i\sigma_j + \tau_i\tau_j\right)\right] \\ &= \sum_{\sigma,\tau} \exp\left[\frac{\beta^2}{2N} \sum_{1\leq i<j\leq N} \left(\sigma_i\sigma_j + \tau_i\tau_j\right)^2\right] \\ &= \sum_{\sigma,\tau} \exp\left[\frac{\beta^2}{N} \sum_{1\leq i<j\leq N} \left(1 + \sigma_i\sigma_j\tau_i\tau_j\right)\right] \\ &= 2^{2N} \exp\left[\frac{\beta^2(N-1)}{2}\right] 2^{-2N} \sum_{\sigma,\tau} \exp\left[\frac{\beta^2}{N} \sum_{1\leq i<j\leq N} \sigma_i\sigma_j\tau_i\tau_j\right] \\ &= 2^{2N} \exp\left[\frac{\beta^2(N-1)}{2}\right] 2^{-2N} \sum_{\sigma,\tau} \exp\left[\frac{\beta^2}{2N}\left(\sum\nolimits_i \sigma_i\tau_i\right)^2 - \frac{\beta^2}{2}\right] \\ &= \mathrm{e}^{-\beta^2/2} 2^{2N} \exp\left[\frac{\beta^2(N-1)}{2}\right] 2^{-2N} \sum_{\sigma,\tau} \exp\left[\frac{\beta^2}{2N}\left(\sum\nolimits_i \sigma_i\tau_i\right)^2\right]. \end{aligned}$$

The σ, τ-sum with the 2^{-2N} in front is just an expectation over two independent coin tossing sequence, and then $\sigma_i\tau_i$ under this measure has just the same distribution as a

single coin tossing. Therefore

$$2^{-2N}\sum_{\sigma,\tau}\exp\left[\frac{\beta^2}{2N}\left(\sum_i\sigma_i\tau_i\right)^2\right]=2^{-N}\sum_{\sigma}\exp\left[\frac{\beta^2}{2N}\left(\sum_i\sigma_i\right)^2\right].$$

This is exactly the partition function of the Curie-Weiss model (see Section 9) with an additional 2^{-N} in front, and β replaced by $\beta^2/2$. Therefore

$$\lim_{N\to\infty}\frac{1}{N}\log 2^{-N}\sum_{\sigma}\exp\left[\frac{\beta^2}{2N}\left(\sum_i\sigma_i\right)^2\right]=\sup_{x\in[-1,1]}\left[\frac{\beta^2}{2}x^2-I\left(x\right)\right]=0$$

where $I\left(x\right)$ is given in (9.2). The last equation is for $\beta^2/2\leq 1/2$, i.e. $\beta\leq 1$. We claim that for $\beta<1$

$$\sup_N 2^{-N}\sum_{\sigma}\exp\left[\frac{\beta^2}{2N}\left(\sum_i\sigma_i\right)^2\right]\leq C\left(\beta\right)<\infty.$$

It can be proved by carefully evaluating Stirling's formula. Another method is to remove the square by an extra Gaussian integration, relying on the fact that

$$\mathrm{e}^{a^2/2}=E\left(\mathrm{e}^{aZ}\right)$$

for a standard Gaussian variable Z, which is evident by completing squares in the exponent

$$E\left(\mathrm{e}^{aZ}\right)=\frac{1}{\sqrt{2\pi}}\int\exp\left[az-z^2/2\right]dz.$$

Therefore, we have[2]

$$\exp\left[\frac{\beta^2}{2N}\left(\sum_i\sigma_i\right)^2\right]=E\left(\exp\left[\frac{\beta}{\sqrt{N}}Z\sum_i\sigma_i\right]\right).$$

The σ-summation can now easily be done individually on the σ_i, leading to

$$\begin{aligned}2^{-N}\sum_{\sigma}\exp\left[\frac{\beta^2}{2N}\left(\sum_i\sigma_i\right)^2\right]&=E\cosh^N\left(\frac{\beta}{\sqrt{N}}Z\right)\\&=E\exp\left[N\log\cosh\left(\frac{\beta}{\sqrt{N}}Z\right)\right].\end{aligned}$$

Now,

$$\frac{d^2\log\cosh\left(x\right)}{dx^2}=1-\tanh^2\left(x\right)\leq 1,$$

and so

$$\log\cosh\left(x\right)\leq x^2/2,$$

[2]The trick is widely used in physics, and sometimes is called "Hubbard-Stratonovich transformation". The physicist Res Jost (1918-1990) used the call it the "Babylonian trick", because the Babylonians invented the method of completing squares.

$$E\exp\left[N\log\cosh\left(\frac{\beta}{\sqrt{N}}Z\right)\right]\leq E\exp\left[\frac{\beta^2}{2}Z^2\right]=\frac{1}{\sqrt{1-\beta^2}}<\infty,$$

if $\beta<1$. Therefore, we have for $\beta<1$

$$\begin{aligned}\mathbb{E}Z_{\beta,0,N}^2&\leq\frac{\mathrm{e}^{-\beta^2/2}}{\sqrt{1-\beta^2}}2^{2N}\exp\left[\frac{\beta^2\left(N-1\right)}{2}\right]\\&=\frac{\mathrm{e}^{-\beta^2/2}}{\sqrt{1-\beta^2}}\left(\mathbb{E}Z_{\beta,0,N}\right)^2.\end{aligned}$$

Let $A_N:=\{Z_N\geq\mathbb{E}Z_N/2\}$. Then

$$\mathbb{E}Z_N=\mathbb{E}\left(Z_N;A_N^c\right)+\mathbb{E}\left(Z_N;A_N\right)\leq\frac{\mathbb{E}Z_N}{2}+\sqrt{\mathbb{E}\left(Z_N^2\right)\mathbb{P}\left(A_N\right)},$$

and therefore

$$\mathbb{P}\left(A_N\right)\geq\frac{\left(\mathbb{E}Z_N\right)^2}{4\mathbb{E}\left(Z_N^2\right)}\geq C\left(\beta\right)>0,$$

i.e.

$$\mathbb{P}\left(\frac{1}{N}\log Z_N\geq\frac{1}{N}\log\mathbb{E}Z_N-\frac{\log 2}{N}\right)\geq C\left(\beta\right).$$

Combining with (3.1), we see that

$$f\left(\beta,0\right)=\lim_{N\to\infty}\frac{1}{N}\mathbb{E}\log Z_N\geq\lim_{N\to\infty}\frac{1}{N}\log\mathbb{E}Z_N=\frac{\beta^2}{4}+\log 2$$

for $\beta<1$. Using Theorem 3.2 d), we conclude that $f\left(\beta,0\right)=\beta^2/4+\log 2$ for $\beta<1$. The same holds true for $\beta=1$ because of the convexity of f which implies continuity, as f is bounded. ■

We will see in Section 3.3 that $f\left(\beta,0\right)<\beta^2/4$ for $\beta>1$. Furthermore, for $h\neq 0$, one has for all β $f\left(\beta,h\right)\neq\lim_{N\to\infty}N^{-1}\log\mathbb{E}Z_N$.

3.3 Guerra's replica symmetric upper bound

The original claim by Sherrington-Kirkpatrick was that

$$f\left(\beta,h\right)=\mathrm{RS}\left(\beta,h\right)\stackrel{\mathrm{def}}{=}\inf_{q\geq 0}\left\{\frac{\left(1-q\right)^2\beta^2}{4}+E_Z\log\cosh\left(h+\beta\sqrt{q}Z\right)+\log 2\right\},$$

where Z is a standard normal random variable, and E_Z here denotes the expectation with respect to Z. We will later see that this is correct for small enough β, but is wrong for large β. It is readily checked that the infimum in q satisfies the following fixed point equation

$$q=E_Z\tanh^2\left(h+\beta\sqrt{q}Z\right).\tag{3.5}$$

For $h=0$, $q=0$ is clearly a solution.

Exercise 3.6
Check that for $h = 0$ and $\beta \leq 1$ the equation (3.5) has the unique solution $q = 0$, and for $\beta > 1$, there is another solution $q(\beta) > 0$ which gives the minimum.

Trickier is the situation for $h \neq 0$.

Lemma 3.7
Let $\beta, h \neq 0$ be arbitrary. Then the equation (3.5) has a unique solution $q(\beta, h) \geq 0$.

The proof is quite tricky and was given by Latala and Guerra. It can be found in Talagrand's book [42]. It is convenient to have this property, but it is not really needed in the proofs that $f(\beta, h) = \mathrm{RS}(\beta, h)$ in certain regions. There are many situations where similar fixed point equations are not known to have unique solutions. As it does not add much to the understanding of the basic problems in spin glasses, I skip the proof here.

Guerra's idea for an upper bound for $f(\beta, h)$ which goes beyond the annealed upper bound, was to try a comparison of the system with SK-Hamiltonian with a simple Hamiltonian with independent spins. The first result was the following remarkable bound:

Theorem 3.8 (Guerra)
For all $\beta > 0$, $h \in \mathbb{R}$, and any N, one has

$$\frac{1}{N}\mathbb{E}\log Z_{\beta,h,N} \leq \mathrm{RS}(\beta, h),$$

and in particular

$$f(\beta, h) \leq \mathrm{RS}(\beta, h).$$

Proof. [3]The proof is again by interpolation quite similar as in (3.2). Let for an arbitrary number $q \geq 0$, (not necessarily the solution of the fixed point equation above), and $t \in [0, 1]$

$$H_t(\sigma) = \beta\sqrt{\frac{t}{N}} \sum_{1\leq i<j\leq N} g_{ij}\sigma_i\sigma_j + \beta\sqrt{1-t}\sum_{i=1}^{N}\sqrt{q}g_i\sigma_i + h\sum_{i=1}^{N}\sigma_i \tag{3.6}$$

where g_i is a set of standard Gaussian variables, independent of the g_{ij}'s.

We write

$$Z_N(t) = \sum_{\sigma}\exp\left[H_t(\sigma)\right],\ \mathcal{G}_t(\sigma) = \frac{\exp\left[H_t(\sigma)\right]}{Z(t)}, \tag{3.7}$$

$$\phi(t) = \frac{1}{N}\mathbb{E}\log Z_N(t). \tag{3.8}$$

Remark that

$$\phi(0) = \int \log\cosh\left(\beta\sqrt{q}x + h\right)\frac{1}{\sqrt{2\pi}}\mathrm{e}^{-x^2/2}dx + \log 2,$$
$$\phi(1) = \frac{1}{N}\mathbb{E}\log Z_{\beta,h,N}$$

[3]The proof was first presented by Francesco Guerra at a conference in Vulcano in 1998.

We again compute the derivative of $\phi(t)$ with respect to t.

$$\begin{aligned}\frac{d\phi(t)}{dt} &= \frac{1}{N}\mathbb{E}\frac{1}{Z(t)}\frac{dZ(t)}{dt} \\ &= \frac{1}{N}\sum_{\sigma\in\Sigma_N}\mathbb{E}\frac{\exp\left[H_t(\sigma)\right]}{Z(t)}\frac{dH_t(\sigma)}{dt}\end{aligned}$$

$$\frac{dH_t}{dt} = \frac{\beta}{2\sqrt{tN}}\sum_{1\leq i<j\leq N} g_{ij}\sigma_i\sigma_j - \frac{\beta\sqrt{q}}{2\sqrt{(1-t)}}\sum_{i=1}^{N} g_i\sigma_i$$

leading to

$$\frac{d\phi}{dt} = S_1 - S_2,$$

where

$$\begin{aligned}S_1 &:= \frac{\beta}{2\sqrt{t}N^{3/2}}\sum_{\sigma\in\Sigma_N}\sum_{1\leq i<j\leq N}\sigma_i\sigma_j\mathbb{E}g_{ij}\frac{\exp\left[H_t(\sigma)\right]}{Z(t)}, \\ S_2 &:= \frac{\beta\sqrt{q}}{2N\sqrt{(1-t)}}\sum_{\sigma\in\Sigma_N}\sum_{i=1}^{N}\sigma_i\mathbb{E}g_i\frac{\exp\left[H_t(\sigma)\right]}{Z(t)}\end{aligned}$$

and as in the section above,

$$S_1 = \frac{\beta^2}{4}\left(1-\nu_t\left(R_N^2\left(\sigma,\sigma'\right)\right)\right),$$

with

$$R_N\left(\sigma,\sigma'\right) = \frac{1}{N}\sum_{i=1}^{N}\sigma_i\sigma_i'.$$

The computation of S_2 is similar. Here one takes in Proposition 1.3, $n=1,\ \Phi(\sigma) = \sigma_i,\ X(\sigma) = g_i$, leading to

$$\begin{aligned}S_2 &= \frac{\beta\sqrt{q}}{2N\sqrt{1-t}}\mathbb{E}\left(\sum_{\sigma}\frac{1}{Z(t)}\sum\nolimits_i g_i\sigma_i\exp\left[H_t(\sigma)\right]\right) \\ &= \frac{\beta\sqrt{q}}{2N\sqrt{1-t}}\sum\nolimits_i \nu_t\left(g_i\sigma_i\right).\end{aligned}$$

$$\nu_t\left(g_i\sigma_i\right) = \beta\sqrt{q}\sqrt{1-t} - \beta\sqrt{q}\sqrt{1-t}\nu_t\left(\sigma_i\sigma_i'\right)$$

$$\begin{aligned}\frac{d\phi}{dt} &= \frac{\beta^2}{4}\nu_t\left\{1-R_N^2-2q\left(1-R_N\right)\right\} \\ &= \frac{\beta^2}{4}\left\{(1-q)^2-\nu_t\left[\left(R_N\left(\sigma,\sigma'\right)-q\right)^2\right]\right\},\end{aligned}$$

which integrated gives

$$\phi(t) - \phi(0) = \frac{\beta^2 t}{4}(1-q)^2 - \frac{\beta^2}{4}\int_0^t \nu_s\left[\left(R_N(\sigma,\tau)-q\right)^2\right] ds, \tag{3.9}$$

and dropping the second summand and taking $t = 1$:

$$\phi(1) - \phi(0) \leq \frac{\beta^2}{4}(1-q)^2 .$$

This implies that for *any* N, we have

$$\frac{1}{N}\mathbb{E}\log Z_{\beta,h,N} \leq \mathrm{RS}(\beta,h) .$$

■

The proof does not only give the desired result, but gives also an expression of the difference, namely

$$\mathrm{RS}(\beta,h) - \frac{1}{N}\mathbb{E}\log Z_{\beta,h,N} = \frac{\beta^2}{4}\int_0^1 \nu_t\left[\left(R_N(\sigma,\tau)-q\right)^2\right] dt \tag{3.10}$$

In order to prove that $f(\beta,h) = \mathrm{RS}(\beta,h)$, one therefore "only" has to show that for the optimal q (i.e. the one given by (3.5)), one has $R_N(\sigma,\tau) \simeq q$ with large $\nu_t^{(2)}$-probability, at least in the t-average. This is not true for large β, but it is true for small β, as we will prove in the next section.

It should also be remarked that Guerra's bound already proves that $f(\beta,0) < \beta^2/4$ for $\beta > 1$. Up to $\beta = 1$, the unique fixed point of (3.5) with $h = 0$ is at $q = 0$ which gives $\mathrm{RS}(\beta,0) = \beta^2/4$ for $\beta \leq 1$, but for $\beta > 1$, there is a fixed point at $q > 0$ which gives a smaller value, so $\mathrm{RS}(\beta,0) < \beta^2/4$ and Guerra's bound proves that $f(\beta,0) \neq f^{\mathrm{ann}}(\beta,0)$, as soon as $\beta > 1$. This was first proved by Comets [20] with a more complicated argument.

3.4 Latala's proof of $f = RS$

The following result was first proved by Talagrand. The simple proof given here is based on an unpublished argument by Latala.

Theorem 3.9
If $\beta \leq 1/2$, then for every h

$$\lim_{N\to\infty}\frac{1}{N}\mathbb{E}\log Z_{\beta,h,N} = \mathrm{RS}(\beta,h) . \tag{3.11}$$

Remark 3.10
In the physics literature, there is a precise prediction about the region for which the above statement should be true which is the celebrated **de Almeida - Thouless condition**

$$\beta^2 E\frac{1}{\cosh^4\left(h+\beta\sqrt{q}Z\right)} \leq 1. \tag{3.12}$$

This is mathematically still not proved, despite the fact that the full Parisi formula for $f(\beta, h)$ is proved. The Parisi formula is analytically very difficult to analyze. On the other hand, it was proved in [16] that a simple iteration scheme for the TAP equations is stable and converges up to and including the AT-line (see the next Section). The TAP equations are closely connected to the free energy via the cavity method. There is also a lot of numerical evidence that (3.12) is the correct condition. (Oral communication by Michel Talagrand). So, I think, there is no reasonable doubt that the AT condition is the precise condition for the validity of (3.11). It is actually rigorously known, that (3.11) is not correct if the de Almeida-Thouless condition is not satisfied. [43].

Latala's argument works only up to $\beta = 1/2$. So, even for $h = 0$, it does not catch the correct critical value.

Proof of Theorem 3.9. The basis of the argument is the representation (3.10). We use the same interpolating Hamiltonian (3.6) and have to apply it to two "replicas", that is we consider the product measure $\mathcal{G}_t^{(2)}$. As usual, we write $\nu_t^{(k)}$ for $\int \mathcal{G}_{t,\omega}^{(k)} \mathbb{P}(d\omega)$, dropping often the index k, and use it also for the expectation. For a deterministic function $f : \Sigma_N^2 \to \mathbb{R}$, we want to compute $d(\nu_t f)/dt$. The f we have in mind is the expression $(R(\sigma, \sigma') - q)^2$. In contrast to the previous section where we investigated the derivative of $\log Z_{N,t}$, we already get an additional replica from the derivative. We write σ^1, σ^2 instead of σ, σ', and write σ^3, σ^4 for additional replicas. The reader should keep in mind that f will always only depend on the first two. We will also assume that f is symmetric: $f(\sigma^1, \sigma^2) = f(\sigma^2, \sigma^1)$ which covers the situation we are interested in.

$$\begin{aligned}
\frac{d}{dt}(\nu_t f) &= \frac{d}{dt}\mathbb{E} \sum_{\sigma^1,\sigma^2} f(\sigma^1, \sigma^2) \frac{\exp\left[H_t(\sigma^1) + H_t(\sigma^2)\right]}{Z(t)^2} \\
&= \mathbb{E} \sum_{\sigma^1,\sigma^2} f(\sigma^1, \sigma^2) \left(H_t'(\sigma^1) + H_t'(\sigma^2)\right) \frac{\exp\left[H_t(\sigma^1) + H_t(\sigma^2)\right]}{Z(t)^2} \\
&- 2 \sum_{\sigma^1,\sigma^2,\sigma^3} f(\sigma^1, \sigma^2) H_t'(\sigma^3) \frac{\exp\left[H_t(\sigma^1) + H_t(\sigma^2) + H_t(\sigma^3)\right]}{Z(t)^3} \\
&= 2\nu_t\left[f(\sigma^1, \sigma^2) H_t'(\sigma^1)\right] - 2\nu_t\left[f(\sigma^1, \sigma^2) H_t'(\sigma^3)\right].
\end{aligned}$$

In the first summand, we have used the fact that f is symmetric, and therefore

$$\nu_t\left[f(\sigma^1, \sigma^2) H_t'(\sigma^1)\right] = \nu_t\left[f(\sigma^1, \sigma^2) H_t'(\sigma^2)\right].$$

All the terms are of the same type as we have already encountered with

$$H_t'(\sigma^1) = \frac{\beta}{2\sqrt{tN}} \sum_{1 \le i < j \le N} g_{ij}\sigma_i^1\sigma_j^1 - \frac{\beta\sqrt{q}}{2\sqrt{(1-t)}} \sum_{i=1}^N g_i\sigma_i^1,$$

so that

$$\begin{aligned}
&\nu_t\left[f\left(\sigma^1,\sigma^2\right)H_t'\left(\sigma^1\right)\right]\\
&=\nu_t\left[f\left(\sigma^1,\sigma^2\right)\left[\frac{\beta}{2\sqrt{tN}}\sum\nolimits_{1\le i<j\le N}g_{ij}\sigma_i^1\sigma_j^1-\frac{\beta\sqrt{q}}{2\sqrt{(1-t)}}\sum\nolimits_{i=1}^{N}g_i\sigma_i^1\right]\right]\\
&=\nu_t\left[f\left(\sigma^1,\sigma^2\right)\frac{\beta^2}{2N}\sum_{1\le i<j\le N}\sigma_i^1\sigma_j^1\left(\sigma_i^1\sigma_j^1+\sigma_i^2\sigma_j^2\right)\right]\\
&-2\nu_t\left[f\left(\sigma^1,\sigma^2\right)\frac{\beta^2}{2N}\sum_{1\le i<j\le N}\sigma_i^1\sigma_j^1\sigma_i^3\sigma_j^3\right]\\
&-\frac{\beta^2q}{2}\nu_t\left[f\left(\sigma^1,\sigma^2\right)\sum_{i=1}^{N}\sigma_i^1\left(\sigma_i^1+\sigma_i^2\right)\right]+2\frac{\beta^2q}{2}\nu_t\left[f\left(\sigma^1,\sigma^2\right)\sum_{i=1}^{N}\sigma_i^1\sigma_i^3\right]\\
&=N\beta^2\nu_t\Big[f\left(\sigma^1,\sigma^2\right)\frac{1+R_{12}^2}{4}-f\left(\sigma^1,\sigma^2\right)\frac{R_{13}^2}{2}\\
&-qf\left(\sigma^1,\sigma^2\right)\frac{1+R_{12}}{2}+qf\left(\sigma^1,\sigma^2\right)R_{13}\Big]
\end{aligned}$$

and similarly

$$\begin{aligned}
\nu_t\left[f\left(\sigma^1,\sigma^2\right)H_t'\left(\sigma^3\right)\right]=N\beta^2\nu_t\Big[&f\left(\sigma^1,\sigma^2\right)\frac{1+2R_{13}^2}{4}-f\left(\sigma^1,\sigma^2\right)\frac{3R_{34}^2}{4}\\
&-qf\left(\sigma^1,\sigma^2\right)\frac{1+2R_{13}}{2}+qf\left(\sigma^1,\sigma^2\right)\frac{3R_{34}}{2}\Big].
\end{aligned}$$

Combining, we get

$$\begin{aligned}
&2\nu_tf\left(\sigma^1,\sigma^2\right)H_t'\left(\sigma^1\right)-2\nu_tf\left(\sigma^1,\sigma^2\right)H_t'\left(\sigma^3\right)\\
&=2N\beta^2\nu_t\Big[f\Big[\frac{1+R_{12}^2}{4}-\frac{R_{13}^2}{2}-q\frac{1+R_{12}}{2}+qR_{13}\\
&-\frac{1+2R_{13}^2}{4}+\frac{3R_{34}^2}{4}+q\frac{1+2R_{13}}{2}-q\frac{3R_{34}}{2}\Big]\Big]\\
&=\frac{1}{2}N\beta^2\nu_t\Big[f\Big[R_{12}^2-4R_{13}^2-2qR_{12}+8qR_{13}\\
&+3R_{34}^2-6qR_{34}\Big]\Big]\\
&=\frac{1}{2}N\beta^2\nu_t\Big[f\Big[\left(R_{12}-q\right)^2-4\left(R_{13}-q\right)^2\\
&+3\left(R_{34}-q\right)^2\Big]\Big]\\
&\le\frac{1}{2}N\beta^2\nu_t\left[f\left[\left(R_{12}-q\right)^2+3\left(R_{34}-q\right)^2\right]\right].
\end{aligned}$$

We apply this inequality to $f\left(\sigma^1,\sigma^2\right):=\exp\left[\lambda N\left(R_{12}-q\right)^2\right]$, $\lambda>0$. For the moment, q is still completely arbitrary. Under $\mathcal{G}_t^{(4)}$, R_{12} and R_{34} are independent, and

therefore

$$\begin{aligned}
&\mathcal{E}_t^{(4)}\left(\left(R_{34}-q\right)^2\exp\left[\lambda N\left(R_{12}-q\right)^2\right]\right)\\
&=\mathcal{E}_t^{(2)}\left(\left(R_{12}-q\right)^2\mathcal{E}_t^{(2)}\left(\exp\left[\lambda N\left(R_{12}-q\right)^2\right]\right)\right)\\
&\leq\mathcal{E}_t^{(2)}\left(\left(R_{12}-q\right)^2\left(\exp\left[\lambda N\left(R_{12}-q\right)^2\right]\right)\right),
\end{aligned}$$

as for $X>0$, X and $\exp[aX]$ are positively correlated for $a>0$. After integrating over $\mathbb{P}$, we get the same inequality for the ν-expectation. Therefore, we get

$$\frac{d}{dt}\nu_t\left(\exp\left[\lambda N\left(R_{12}-q\right)^2\right]\right)\leq 2N\beta^2\nu_t\left(\left(R_{12}-q\right)^2\left(\exp\left[\lambda N\left(R_{12}-q\right)^2\right]\right)\right).$$

As a consequence, we conclude that

$$\frac{d}{dt}\nu_t\left(\exp\left[\left(\lambda-2t\beta^2\right)N\left(R_{12}-q\right)^2\right]\right)\leq 0 \tag{3.13}$$

for $t\leq\lambda/2\beta^2$.

The crucial point with choosing q just comes now. We investigate exponential moments of $N\left(R_{12}-q\right)^2$ under ν_0. Under $\mathcal{G}_{t=0}$ the single spins are independent, and distributed according to

$$\mathcal{G}_{t=0}\left(\sigma_i=1\right)=\frac{\exp\left[\beta\sqrt{q}g_i+h\right]}{\cosh\left(\beta\sqrt{q}g_i+h\right)}.$$

Therefore

$$\mathcal{E}_{t=0}\left(\sigma_i\right)=\tanh\left(\beta\sqrt{q}g_i+h\right),$$

and

$$\mathcal{E}_{t=0}^{(2)}\left(\sigma_i^1\sigma_i^2\right)=\tanh^2\left(\beta\sqrt{q}g_i+h\right).$$

Therefore,

$$v_0\left(\sigma_i^1\sigma_i^2\right)=E_Z\tanh^2\left(\beta\sqrt{q}Z+h\right).$$

If q is the solution of the fixed point equation (3.5), then this is q. It is therefore clear that R_{12} concentrates around q under ν_0. The $\sigma_i^1\sigma_i^2$ are under ν_0 i.i.d. ±1 random variables with expectation q if the fixed point equation is satisfied. Some elementary computation for the binomial distribution then lead easily to the fact that for $\lambda<1/2$, q the solution of (3.5) and all N

$$\nu_0\exp\left[\lambda N\left(R_{12}-q\right)^2\right]\leq\left(1-2\lambda\right)^{-1}. \tag{3.14}$$

If $\beta<1/2$, then $\kappa\left(\beta\right):=1/4-\beta^2>0$, and with $\lambda:=1/2-\kappa$, we have $\lambda-2\beta^2=\kappa>0$. Using (3.14), together with (3.13) leads to

$$\nu_t\exp\left[\kappa\left(\beta\right)N\left(R_{12}-q\right)^2\right]\leq\frac{1}{\sqrt{\kappa\left(\beta\right)}}<\infty$$

for all $N \in \mathbb{N},\ t \leq 1$, and $\beta < 1/2$.

Together with (3.9), this immediately proves the Theorem 3.9. ∎

In fact, the exponential estimate in the proposition evidently implies that

$$\nu_t \left((R_{12} - q)^2 \right) \leq \frac{C(\beta)}{N},$$

so that we obtain much more, namely

$$\left| \frac{1}{N} \mathbb{E} \log Z_{\beta,h,N} - \mathrm{RS}(\beta, h) \right| \leq \frac{C(\beta)}{N}.$$

3.5 On the TAP equations

The appearance of q and the fixed point equation is a bit mysterious, but the form of the second summand of (3.6) indicates that the σ_i under the Gibbs measure in the high temperature (small β) regime essentially look like independent ones with a random expectation. This is indeed the case and has been proved by Talagrand in [42].

The Gibbs expectations $m_i(\omega) := \mathcal{E}_\omega \sigma_i$ satisfy the the TAP equations (for Thouless, Anderson and Palmer). These equations are somewhat similar to the mean field equation (9.3) for the Curie-Weiss model

$$m = \tanh(h + \beta m),$$

but they are more tricky. (We drop β in front of h in accordance with our habit in the SK-model).

We first give a heuristic derivation of the TAP equations. Let's try to compute m_1 in terms of the others:

$$m_1 = \frac{\sum_\sigma \sigma_1 \exp\left[\frac{\beta}{\sqrt{N}} \sum_{i<j} g_{ij} \sigma_i \sigma_j + h \sum_{i=1}^N \sigma_i \right]}{\sum_\sigma \exp\left[\frac{\beta}{\sqrt{N}} \sum_{i<j} g_{ij} \sigma_i \sigma_j + h \sum_{i=1}^N \sigma_i \right]}.$$

We can split the Hamiltonian by taking everything which depends on σ_1 outside: With $y_{1,\sigma^{(1)}} := N^{-1/2} \sum_{j=2}^N g_{1j} \sigma_j$ for the so-called cavity variables, and where $\sigma^{(1)} := (\sigma_2, \ldots, \sigma_N)$.

$$\frac{\beta}{\sqrt{N}} \sum_{i<j} g_{ij} \sigma_i \sigma_j + h \sum_{i=1}^N \sigma_i = \sigma_1 \left[\beta y_{1,\sigma^{(1)}} + h \right] + \frac{\beta}{\sqrt{N}} \sum_{2 \leq i < j \leq N} g_{ij} \sigma_i \sigma_j + h \sum_{i=2}^N \sigma_i.$$

The second part is just the Hamiltonian for $\sigma_2, \ldots, \sigma_N$ where all the interactions with σ_1 are dropped. We denote the corresponding Gibbs expectation as $\mathcal{E}^{\mathrm{cut}(1)}$.

Summing out σ_1, one obtains

$$m_1 = \frac{\mathcal{E}^{\mathrm{cut}(1)} \sinh\left(\beta y_{1,\sigma^{(1)}} + h \right)}{\mathcal{E}^{\mathrm{cut}(1)} \cosh\left(\beta y_{1,\sigma^{(1)}} + h \right)},$$

The curious fact is that one can take $\mathcal{E}^{\mathrm{cut}(1)}$ inside sinh and cosh:

$$\frac{\mathcal{E}^{\mathrm{cut}(1)} \sinh\left(\beta y_{1,\sigma^{(1)}} + h\right)}{\mathcal{E}^{\mathrm{cut}(1)} \cosh\left(\beta y_{1,\sigma^{(1)}} + h\right)} \approx \frac{\sinh\left(\beta \mathcal{E}^{\mathrm{cut}(1)} y_{1,\sigma^{(1)}} + h\right)}{\cosh\left(\beta \mathcal{E}^{\mathrm{cut}(1)} y_{1,\sigma^{(1)}} + h\right)}$$
$$= \tanh\left(h + \frac{\beta}{\sqrt{N}} \sum_{j=1}^{2} g_{1j} m_j^{(1)}\right),$$

where $m_j^{(1)} := \mathcal{E}^{\mathrm{cut}(1)} \sigma_j$. There is no mystery in the above computation. In fact, one cannot just take the $\mathcal{E}^{\mathrm{cut}(1)}$ expectation inside sinh and cosh, but one can do it up to a factor which cancels out.[4]

In order to see this, write

$$y_{1,\sigma^{(1)}} = \frac{1}{\sqrt{N}} \sum_{j=2}^{N} g_{1j} \left(\sigma_j - m_j^{(1)}\right) + \frac{1}{\sqrt{N}} \sum_{j=2}^{N} g_{1j} m_j^{(1)},$$

where now $m_i^{(1)}$ denotes the Gibbs expectation of σ_i under $\mathcal{E}^{\mathrm{cut}(1)}$. One should now remark that $\left\{m_j^{(1)}\right\}$ and $\{g_{1j}\}$ are independent random variables. We pretend now that the variables $\left\{\sigma_j - m_j^{(1)}\right\}_{2 \le j \le N}$ are sufficiently independent under $\mathcal{G}^{\mathrm{cut}(1)}$ that they satisfy a central limit theorem, and assuming that this is correct, it is then easy to see that $\frac{1}{\sqrt{N}} \sum_{j=2}^{N} g_{1j} \left(\sigma_j - m_j^{(1)}\right)$ satisfies a CLT, too. The variance γ^2 does not interest us. Anyway, accepting these somewhat dubious facts, one would get

$$\mathcal{E}^{\mathrm{cut}(1)} \sinh\left(\beta y_{1,\sigma^{(1)}} + h\right) \approx \frac{1}{2} \exp\left[\frac{\beta}{\sqrt{N}} \sum\nolimits_{j=2}^{N} g_{1j} m_j^{(1)} + h\right] \mathrm{e}^{\gamma^2/2}$$
$$- \frac{1}{2} \exp\left[-\frac{\beta}{\sqrt{N}} \sum\nolimits_{j=2}^{N} g_{1j} m_j^{(1)} - h\right] \mathrm{e}^{\gamma^2/2},$$

and similarly, with $\mathcal{E}^{\mathrm{cut}(1)} \cosh\left(\beta y_{1,\sigma^{(1)}} + h\right)$. Therefore, the $\mathrm{e}^{\gamma^2/2}$ factor cancels out, and

$$m_1 \approx \tanh\left(\frac{\beta}{\sqrt{N}} \sum\nolimits_{j=2}^{N} g_{1j} m_j^{(1)} + h\right).$$

Similarly, one can do that with any m_i :

$$m_i \approx \tanh\left(\frac{\beta}{\sqrt{N}} \sum\nolimits_{j:j\neq i} g_{ij} m_j^{(i)} + h\right), \tag{3.15}$$

where $m_j^{(i)} := \mathcal{E}^{\mathrm{cut}(i)}(\sigma_j)$, and $g_{ij} = g_{ji}$ for $i > j$. This is one form of the TAP equations. It is clear that they should hold only in an approximate sense as $N \to \infty$, as we have

[4] This is an artifact of some very special properties of the SK model. In only slightly more complicated models, for instance if the σ_i take more than two possible values, this is no longer the case, and the TAP equations become more complicated.

used a CLT argument. Typically, one writes them in a form where one replaces $m_j^{(i)}$ by m_j. It turns out that the correction is for each j only of order $1/\sqrt{N}$ but this contributes to the outcome. We sketch the argument. Using the TAP equation for the m_j we have

$$m_j \approx \tanh\left(\frac{\beta}{\sqrt{N}}\sum\nolimits_{k:k\neq j} g_{jk}m_k^{(j)} + h\right).$$

On the other hand, $m_j^{(i)}$ is obtained by just cutting the interaction between i and j :

$$m_j^{(i)} \approx \tanh\left(\frac{\beta}{\sqrt{N}}\sum\nolimits_{k:k\neq j,i} g_{jk}m_k^{(j)} + h\right).$$

Therefore, by Taylor, as $\tanh'(x) = 1 - \tanh^2(x)$

$$m_j \approx m_j^{(i)} + \frac{\beta}{\sqrt{N}} g_{ji} m_i^{(j)} \left(1 - m_j^{(i)2}\right)$$

The correction is of order $1/\sqrt{N}$, and we are not interested in corrections of lower order.

One should also observe that for $N^{-1}\sum_{i=1}^N m_i^2$, the correction is irrelevant. If one believes that there should be a LLN for this quantity, one gets from (3.15) that

$$q = \lim_{N\to\infty} \frac{1}{N}\sum_{i=1}^N m_i^2 = E_Z \tanh^2\left(h + \sqrt{q}\beta Z\right),$$

that implies that $q = q(\beta, h)$ satisfies exactly our fixed point equation (3.5). We next implement the corrections for $m_j^{(i)}$ into (3.15). We then obtain

$$m_i \approx \tanh\left(h + \frac{\beta}{\sqrt{N}}\sum_{j:j\neq i} g_{ij}m_j - \beta^2 m_i \frac{1}{N}\sum_{j:j\neq i} g_{ij}^2\left(1 - m_j^{(i)2}\right)\right).$$

$$\frac{1}{N}\sum_{j:j\neq i} g_{ij}^2\left(1 - m_j^{(i)2}\right) \approx \frac{1}{N}\sum_{j:j\neq i}\left(1 - m_j^{(i)2}\right) \approx 1 - q.$$

Therefore,

$$m_i \approx \tanh\left(h + \frac{\beta}{\sqrt{N}}\sum_{j:j\neq i} g_{ij}m_j - (1-q)\beta^2 m_i\right). \tag{3.16}$$

This is the TAP equation in the usual form. The correction term $-(1-q)\beta^2 m_i$ is usually called "Onsager correction", but it is there for the very same reason as the correction term in Itô calculus, so I usually call it "Onsager-Itô-correction".

In physics literature, the equations are claimed to be correct also in the low temperature regime, but there, m_i is not the (global) Gibbs mean of the spin variables, but rather the average under "pure states", whatever that exactly means.[5]

[5] Despite of the recent progress on the low-temperature SK-model, there are absolutely no rigorous results on the validity of the TAP equations in low temperature.

The equation has some precise global stability property exactly up to the AT-line. This has been discussed in my recent paper [16]. One can define what a solution means (in a $N \to \infty$ sense) without any reference to the SK model. The way it was set up was via an iterative construction. For every N one defines a sequence $\left\{m_i^{[k]}\right\}_{1\leq i\leq N}$, $k \geq 0$, by

$$m_i^{[0]} := 0,\ m_i^{[1]} = \sqrt{q},$$

$$m_i^{[k+1]} := \tanh\left(h + \frac{\beta}{\sqrt{N}} \sum_{j:j\neq i} g_{ij} m_j^{[k]} - (1-q)\,\beta^2 m_i^{[k-1]}\right),\ k \geq 1.$$

Then

Theorem 3.11

$$\limsup_{k,l\to\infty} \limsup_{N\to\infty} \frac{1}{N} \sum_{i=1}^{N} \left(m_i^{[l]} - m_i^{[k]}\right)^2 = 0,\ \text{a.s.}$$

if and only if the AT-condition is satisfied.

As the construction leads to an interesting representation and has lead to developments in other fields (see [8]), we sketch it.

In order to keep the formulas short, we use the abbreviation

$$\mathrm{Th}\,(x) = \tanh\,(h + \beta x)\,.$$

Then

$$m_i^{[2]} = \mathrm{Th}\left(\sqrt{q}\xi_i^{[1]}\right)$$

where

$$\xi_i^{[1]} = \frac{1}{\sqrt{N}} \sum_{j=1}^{N} g_{ij},$$

and

$$m_i^{[3]} = \mathrm{Th}\left(\frac{1}{\sqrt{N}} \sum\nolimits_{j=1}^{N} g_{ij} m_j^{[2]} - \beta\,(1-q)\,\sqrt{q}\right). \tag{3.17}$$

We first discuss now $m^{[3]}$ carefully, and then sketch the general scheme for the higher order iterates. A seeming difficulty in the analysis is that $m^{[2]}$ depends on the g_{ij} in a non-linear way. However, it turns out that this can be analyzed in a simple way. For that, we "correct" the matrix (g_{ij}) and make it independent of the $\xi^{[1]}$ and therefore independent of the $m^{[2]}$. As the $\xi^{[1]}$ are linear combinations of the matrix elements, this can be done in a straightforward way. The exact formula is a bit complicated, but

$$g_{ij}^{[2]} := g_{ij} - \frac{\xi_i^{[1]} + \xi_j^{[1]}}{\sqrt{N}}$$

is "sufficiently" independent of $\xi^{[1]}$. Just check the covariances: Whereas

$$\mathbb{E}\left(\xi_i^{[1]} g_{ij}\right) = \frac{1}{\sqrt{N}},$$

we have

$$\mathbb{E}\left(\xi_i^{[1]} g_{ij}^{[2]}\right) = 2N^{-3/2}.$$

For the sketchy approach done here, it suffices to work with $\left(g_{ij}^{[2]}\right)$ as defined above, but one should be aware that in order to get $g^{[2]}$ fully independent of the $\xi^{[1]}$, one needs a more complicated expression. If we substitute $g^{[2]}$ for the g_{ij} in (3.17), we get

$$\begin{aligned} m_i^{[3]} = \mathrm{Th}\Big(\frac{1}{\sqrt{N}} \sum\nolimits_{j=1}^{N} g_{ij}^{[2]} m_j^{[2]} + \xi_i^{[1]} \frac{1}{N} \sum\nolimits_{j=1}^{N} m_j^{[2]} \\ + \frac{1}{N} \sum\nolimits_{j=1}^{N} \xi_j^{[1]} m_j^{[2]} - \beta\,(1-q)\,\sqrt{q}\Big). \end{aligned}$$

By the law of large numbers

$$\frac{1}{N} \sum_{j=1}^{N} m_j^{[2]} = \frac{1}{N} \sum_{j=1}^{N} \mathrm{Th}\left(\sqrt{q}\xi_i^{[1]}\right) \approx \int \mathrm{Th}\left(\sqrt{q}x\right) \phi\left(dx\right),$$

where ϕ is the standard normal distribution. We set $\gamma_1 := \int \mathrm{Th}\left(\sqrt{q}x\right) \phi\left(dx\right)$.

$$\begin{aligned} \frac{1}{N} \sum\nolimits_{j=1}^{N} \xi_j^{[1]} m_j^{[2]} &= \frac{1}{N} \sum\nolimits_{j=1}^{N} \xi_j^{[1]} \mathrm{Th}\left(\sqrt{q}\xi_i^{[1]}\right) \\ &\approx \int x\,\mathrm{Th}\left(\sqrt{q}x\right) \phi\left(dx\right) = \beta\sqrt{q}\,(1-q), \end{aligned}$$

which cancels the Onsager term. Therefore, we get

$$m_i^{[3]} \approx \mathrm{Th}\left(\frac{1}{\sqrt{N}} \sum\nolimits_{j=1}^{N} g_{ij}^{[2]} m_j^{[2]} + \gamma_1 \xi_i^{[1]}\right). \tag{3.18}$$

We don't give precise estimates for the approximations involved. This is indeed quite a delicate point in the later iterations, and it contributed to the length of [16], but we don't wish to discuss this point here, as it would become too lengthy, and somewhat obscures the basic simplicity of the structure.

The above expression reveals a simple structure of $m^{[3]}$: The $\xi_i^{[1]}$ are of course Gaussian with variance $1 - 1/N \approx 1$, and are nearly independent, as

$$\mathbb{E}\xi_i^{[1]}\xi_j^{[1]} = \frac{1}{N},\ i \neq j.$$

The first summand inside $\mathrm{Th}\left(\cdot\right)$ in (3.18) is of course *not* Gaussian, but it is Gaussian *conditioned* on $\mathcal{F}_1 := \sigma\left(\xi_i^{[1]} : 1 \leq i \leq N\right)$. We compute the conditional variance. First remark that

$$\frac{1}{\sqrt{N}} \sum\nolimits_{j=1}^{N} g_{ij}^{[2]} m_j^{[2]} \approx \frac{1}{\sqrt{N}} \sum\nolimits_{j=1}^{N} g_{ij}^{[2]} \left(m_j^{[2]} - \gamma_1\right)$$

as $\sum_j g_{ij}^{[2]} \approx 0$ by construction. (Actually, one gets exactly 0 if one corrects the g_{ij} precisely such that they become independent of the $\xi^{[1]}$). If one chooses i.i.d. copies of the $\xi_i^{[1]}$, independent of everything defined so far, call them $\bar{\xi}_i$, then

$$\bar{g}_{ij} = g_{ij}^{[2]} + \frac{\bar{\xi}_i + \bar{\xi}_j}{\sqrt{N}}$$

has the same distribution as the original g_{ij} and is independent of the the $\xi^{[1]}$. Furthermore

$$\frac{1}{\sqrt{N}} \sum\nolimits_{j=1}^{N} \frac{\bar{\xi}_i + \bar{\xi}_j}{\sqrt{N}} \left(m_j^{[2]} - \gamma_1 \right) \approx 0.$$

Therefore, we see that the conditional distribution of

$$\frac{1}{\sqrt{N}} \sum\nolimits_{j=1}^{N} g_{ij}^{[2]} m_j^{[2]} \tag{3.19}$$

is centered Gaussian with variance

$$\frac{1}{N} \sum_{j=1}^{N} \left(m_j^{[2]} - \gamma_1 \right)^2 = \frac{1}{N} \sum_{j=1}^{N} m_j^{[2]2} - \gamma_1^2 \approx q - \gamma_1^2.$$

Therefore, although the unconditional distribution is for finite N clearly not Gaussian, it is approximately so, because the conditional variance is by the LLN, in the $N \to \infty$ limit, a constant. Furthermore, it becomes asymptotically independent of $\xi^{[1]}$.

We describe now the construction for general k, and give an outline of the proof. $m_i^{[k]}$ has the following representation:

$$m_i^{[k]} \approx \mathrm{Th} \left(\frac{1}{\sqrt{N}} \sum\nolimits_j g_{ij}^{[k-1]} m_j^{[k-1]} + \sum\nolimits_{t=1}^{k-2} \gamma_t \xi_i^{[t]} \right) \tag{3.20}$$

with real coefficients γ_t, random variables $\xi_i^{[t]}$, and transformed matrices $\left(g_{ij}^{[k-1]} \right)$ which we will describe. $g_{ij}^{[1]} := g_{ij}$, and $g^{[2]}$ we have defined already above. To define them, let $\langle \cdot, \cdot \rangle$ be the inner product in $\mathbb{R}^N$ which is the standard one, divided by N. Π_k is the orthogonal projection in $\mathbb{R}^N$, with respect to this inner product, onto the subspace spanned by the vectors 1, $m^{[1]}, \ldots, m^{[k]}$. Let

$$\phi^{[k]} = \frac{m^{[k]} - \Pi_{k-1} \left(m^{[k]} \right)}{\left\| m^{[k]} - \Pi_{k-1} \left(m^{[k]} \right) \right\|},$$

where $\|x\| := \sqrt{\langle x, x \rangle}$, $x \in \mathbb{R}^N$. $\phi^{[1]}$ is the vector identically to 1.

The representation (3.20) leads to an evaluation of the inner products $\left\langle m^{[s]}, m^{[t]} \right\rangle$ in the form

$$\begin{aligned} \lim_{N \to \infty} \left\langle m^{[t]}, m^{[t]} \right\rangle &= q, \ \forall t \\ \lim_{N \to \infty} \left\langle m^{[s]}, m^{[t]} \right\rangle &= \rho_s, \ s < t, \end{aligned} \tag{3.21}$$

where the sequence $\{\rho_s\}$ is related to the sequence $\{\gamma_t\}$, as will be explained below. It is important that $\lim_{N\to\infty}\langle m^{[s]}, m^{[t]}\rangle$ does not depend on t provided $t > s$.

The $\xi_i^{[1]}$ we have defined above. They are Gaussian's, and for $N \to \infty$, essentially i.i.d. standard. The exact construction of the random variables and constants entering (3.20) is intertwined. The crucial point is the construction of the $g^{[t]}$ where $g_{ij}^{[1]} = g_{ij}$. Given these, the $\xi^{[t]}$ are defined by

$$\xi_i^{[t]} := \frac{1}{\sqrt{N}} \sum_{j=1}^{N} g_{ij}^{[t]} \phi_j^{[t]}. \tag{3.22}$$

We define $\mathcal{F}_t$ as the σ-field generated by the $\xi^{[s]},\ s \le t$.

Next the recursive construction of $g^{[t]}$. It is done such that $g^{[t]}$ is conditionally Gaussian given $\mathcal{F}_{t-2}$, and conditionally independent of $\mathcal{F}_{t-1}$. We describe now, how to construct $g^{[t+1]}$: On the RHS of (3.22), the $\phi_j^{[t]}$ are $\mathcal{F}_{t-1}$-m.b., and the $g_{ij}^{[t]}$ are, conditioned on $\mathcal{F}_{t-2}$ Gaussians which are independent of the $\phi^{[t]}$. We can therefore correct the $g_{ij}^{[t]}$ by linear combinations of the $\phi_j^{[t]}$. These corrected matrix we call $g_{ij}^{[t+1]}$. It is evidently Gaussian, conditioned on $\mathcal{F}_{t-1}$, and by construction, conditionally independent of $\mathcal{F}_t$. The exact expression is unfortunately slightly involved, but as before with $g^{[2]}$, we use a simplified formula which gives asymptotically as $N \to \infty$ the correct expression

$$g_{ij}^{[t+1]} = g_{ij}^{[t]} - \frac{\xi_i^{[t]}\phi_j^{[t]} + \xi_j^{[t]}\phi_i^{[t]}}{\sqrt{N}}.$$

(For $t = 1$, this is the old expression as $\phi_i^{[1]} = 1$). It is readily checked by this construction that

$$\frac{1}{\sqrt{N}} \sum_{j=1}^{N} g_{ij}^{[t]} \phi_j^{[s]} \approx 0$$

for $s < t$, and therefore

$$\begin{aligned}\frac{1}{\sqrt{N}} \sum_{j=1}^{N} g_{ij}^{[t]} m_j^{[t]} &\approx \left\| m^{[t]} - \Pi_{k-1}\left(m^{[t]}\right)\right\| \frac{1}{\sqrt{N}} \sum_{j=1}^{N} g_{ij}^{[t]} \phi_j^{[t]} \\ &= \left\| m^{[t]} - \Pi_{k-1}\left(m^{[t]}\right)\right\| \xi_i^{[t]}.\end{aligned}$$

Furthermore, in the same way as in the analysis of (3.19), one gets, that conditionally on $\mathcal{F}_{t-1}$, $N^{-1/2}\sum_{j=1}^{N} g_{ij}^{[t]}\phi_j^{[t]}$ has the same distribution as $N^{-1/2}\sum_{j=1}^{N} \bar{g}_{ij}\phi_j^{[t]}$ with copies $\bar{g}_{ij}$ of g_{ij} which are independent of $\mathcal{F}_{t-1}$.

$$\frac{1}{\sqrt{N}} \sum_{j=1}^{N} g_{ij}^{[t]} m_j^{[t]}$$

is Gaussian, as $N \to \infty$, with variance

$$\left\| m^{[t]} - \Pi_{t-1}\left(m^{[t]}\right)\right\|^2 = \left\|m^{[t]}\right\|^2 - \left\|\Pi_{t-1}\left(m^{[t]}\right)\right\|^2.$$

We will see that

$$\lim_{N\to\infty} \left\|m^{[t]}\right\|^2 = q, \ \lim_{N\to\infty} \left\|\Pi_{t-1}\left(m^{[t]}\right)\right\|^2 = \Gamma_{t-1}^2, \tag{3.23}$$

where

$$\Gamma_{t-1}^2 := \sum_{j=1}^{t-1} \gamma_j^2.$$

We have however not yet explained how the coefficients γ_j are constructed. Let us first explain that the validity of (3.20) and (3.23) leads to an evaluation of the inner products $\left\langle m^{[k]}, m^{[s]}\right\rangle$.

Assume that (3.20) is valid for k. The considerations above reveal that in the limit as $N \to \infty$, the expression inside $\mathrm{Th}\,(\cdot)$ is a sum of independent Gaussians, the $\xi_i^{[t]}$ with variance 1, and $N^{-1/2}\sum_{j=1}^N g_{ij}^{[k-1]} m_j^{[k-1]}$ with variance $q - \Gamma_{k-2}^2$. From that, we get

$$\lim_{N\to\infty} \left\|m^{[k]}\right\|^2 = q.$$

For $s < k$, we get that

$$\begin{aligned}\lim_{N\to\infty} \left\langle m^{[k]}, m^{[s]}\right\rangle = E\,\mathrm{Th}\left(\sqrt{q-\Gamma_{s-1}^2}Z'' + \gamma_{s-1}Z' + \Gamma_{s-2}Z\right)\\ \times\,\mathrm{Th}\left(\sqrt{q-\Gamma_{s-2}^2}Z' + \Gamma_{s-2}Z\right)\end{aligned}$$

with independent standard Gaussians Z, Z', Z''.

Define the function $\psi : [0,q] \to [0,q]$ by

$$\psi\,(t) := E\,\mathrm{Th}\left(\sqrt{t}Z + \sqrt{q-t}Z'\right)\mathrm{Th}\left(\sqrt{t}Z + \sqrt{q-t}Z''\right)$$

again with independent Z, Z', Z''. Remark that $\psi\,(0) = \left(E\,\mathrm{Th}\left(\sqrt{q}Z\right)\right)^2 = \gamma_1^2$, and $\psi\,(q) = q$. A simple computation gives

$$\begin{aligned}E\,\mathrm{Th}\left(\sqrt{q-\Gamma_{s-1}^2}Z'' + \gamma_{s-1}Z' + \Gamma_{s-2}Z\right)\mathrm{Th}\left(\sqrt{q-\Gamma_{s-2}^2}Z' + \Gamma_{s-2}Z\right)\\ = \psi\left(\gamma_{s-1}\sqrt{q-\Gamma_{s-2}^2} + \Gamma_{s-2}^2\right).\end{aligned}$$

So, we see that the relation between the ρ's and the γ's is given as

$$\rho_s = \psi\left(\gamma_{s-1}\sqrt{q-\Gamma_{s-2}^2} + \Gamma_{s-2}^2\right).$$

We will see that actually the expression inside $\psi(\cdot)$ is ρ_{s-1}, i.e.

$$\gamma_n = \frac{\rho_n - \Gamma_{n-1}^2}{\sqrt{q - \Gamma_{n-1}^2}}, \tag{3.24}$$

but for the moment, these relations are a bit of a miracle. This is now solved by boosting the crucial relation (3.20). Assume that is valid for k, and we sketch the argument how to prove it for $k+1$ instead of k assuming that it is valid on smaller levels. For that, we start with the original definition

$$m_i^{[k+1]} = \mathrm{Th}\left(\frac{1}{\sqrt{N}}\sum_j g_{ij} m_j^{[k]} - \beta(1-q)\, m_i^{[k-1]}\right).$$

Replacing $g = g^{[1]}$ by $g^{[2]}$ gives

$$\begin{aligned} m_i^{[k+1]} = \mathrm{Th}\Big(&\frac{1}{\sqrt{N}}\sum\nolimits_{j=1}^N g_{ij}^{[2]} m_j^{[k]} + \xi_i^{[1]}\frac{1}{N}\sum\nolimits_{j=1}^N m_j^{[k]} \\ &+ \frac{1}{N}\sum\nolimits_{j=1}^N \xi_j^{[1]} m_j^{[k]} - \beta(1-q)\sqrt{q}\Big). \end{aligned}$$

Given the representation (3.20) for k, we get

$$\frac{1}{N}\sum\nolimits_{j=1}^N m_j^{[k]} \approx \gamma_1,$$

and for $k \geq 3$

$$\begin{aligned} \frac{1}{N}\sum\nolimits_{j=1}^N \xi_j^{[1]} m_j^{[k]} &\approx EZ\,\mathrm{Th}\left(\sqrt{q-\gamma_1^2}Z' + \gamma_1 Z\right) \\ &= \beta(1-q)\,\gamma_1, \end{aligned}$$

so that

$$m_i^{[k+1]} \approx \mathrm{Th}\left(\frac{1}{\sqrt{N}}\sum\nolimits_{j=1}^N g_{ij}^{[2]} m_j^{[k]} + \gamma_1\xi_i^{[1]} - \beta(1-q)\left(m_i^{[k-1]} - \gamma_1\right)\right).$$

Based on the inductive use of (3.20), it is not difficult to check that by the sequence of replacements $g^{[2]} \to g^{[3]} \to \cdots \to g^{[k]}$ we successively produce the terms $\gamma_2\xi^{[2]}$, $\gamma_3\xi^{[3]}, \ldots,$ $\gamma_{k-1}\xi^{[k-1]}$, and "eat up" successively the Onsager term with $\beta(1-q)$, where the γ's are given recursively by (3.24).

All this is correct for any parameter β, and also in the low temperature regime. However, it is useful only if

$$\lim_{N\to\infty} \mathrm{var}\left(\frac{1}{\sqrt{N}}\sum\nolimits_j g_{ij}^{[k-1]} m_j^{[k-1]}\right) = q - \sum_{j=1}^{\infty}\gamma_j^2 = 0.$$

The reader will have no difficult to check that this is true if and only if

$$\lim_{k\to\infty} \rho_k = q. \tag{3.25}$$

It is elementary to check that the function ψ is strictly convex, and as it has q as a fixed point and satisfies $\psi(0) = \gamma_1 > 0$ if $h > 0$, we get that (3.25) holds true if and only if $\psi'(q) \leq 1$. By an elementary computation, this is equivalent to the AT-condition. So, the Theorem 3.11 follows.

In principle, TAP type equations can be discussed for most of the mean-field spin glasses, for instance also for the perceptron. The details of an iterative scheme have however not been worked out in other cases besides SK. Even more important would be a discussion of the TAP equations in low temperature.

4 Ruelle's probability cascades

4.1 The Poisson-Dirichlet point process

We have already introduced the Poisson Dirichlet point process $\mathrm{PD}(\zeta)$ with parameter $0 < \zeta < 1$. It is obtained from a PPP $\left(\zeta x^{-\zeta-1}dx\right)$ $\{\eta_i\}$ on $\mathbb{R}^+$ via the normalization $\bar{\eta}_i := \eta_i / \sum_j \eta_j$, see Definition 2.10. $\zeta < 1$ implies $\sum_j \eta_j < \infty$ almost surely. If Ξ is a point process on $\mathbb{R}^+$ where the points have a finite sum, we write $\mathcal{N}(\Xi)$ for this normalization.

As remarked, the point processes don't care for the labeling of the points by the natural numbers. Such a labeling can always be done by ordering the points downwards if there exists a largest point.

Proposition 4.1

a) Let $\{\eta_i\}$ be a PPP $\left(\zeta x^{-\zeta-1}dx\right)$ on $\mathbb{R}^+$, and let $\{Y_i\}$ be an i.i.d. sequence of real-valued random variables with distribution μ. Assume that $\psi : \mathbb{R} \to \mathbb{R}$ is a measurable function with $C(\zeta) := \int \mathrm{e}^{\zeta\psi(y)}\mu(dy) < \infty$. Then the point process $\left\{\left(\mathrm{e}^{\psi(Y_i)}\eta_i, Y_i\right)\right\}_i$ on $\mathbb{R}^+ \times \mathbb{R}$ has the same law as the point process $\left\{\left(C(\zeta)^{1/\zeta}\eta_i, Y_i'\right)\right\}_i$ where Y_i' is i.i.d. with distribution

$$\mu'(dy) := C(\zeta)^{-1}\mathrm{e}^{\zeta\psi(y)}\mu(dy).$$

b) If $\{\eta_i\}_i$ is a PPP $\left(\zeta x^{-\zeta-1}dx\right)$, and $0 < \zeta < \zeta' < 1$, then $\left\{\eta_i^{\zeta'}\right\}_i$ is a PPP $\left(\frac{\zeta}{\zeta'}x^{-\zeta/\zeta'-1}dx\right)$

c) Let $\Xi_k = \left\{\eta_i^k\right\}_i$, $k \in \mathbb{N}$ be an i.i.d. sequence of PPP $\left(\zeta x^{-\zeta-1}dx\right)$'s, and let $\{y_k\}$ be a sequence of positive real numbers satisfying $C(\zeta) \stackrel{\mathrm{def}}{=} \sum_k y_k^\zeta < \infty$. Then the point process $\left\{y_k\eta_i^k / C(\zeta)^{1/\zeta}\right\}_{i,k}$ is also a PPP $\left(\zeta x^{-\zeta-1}dx\right)$.

d) Let $0 < \zeta < \zeta' < 1$, $\mathbf{y} = \{y_i\}$ *be a sequence in* $\mathbb{R}^+$ *with* $C(\zeta, \mathbf{y}) \stackrel{\text{def}}{=} \sum_k y_k^\zeta < \infty$, *and* $\Xi_k = \left\{\eta_i^k\right\}_i$, $k \in \mathbb{N}$ *be an i.i.d. sequence of* PPP $\left(\zeta x^{-\zeta-1}dx\right)$*'s. Then*

$$\sum_{i,k} \left(y_k \eta_i^k\right)^{\zeta'} < \infty,$$

almost surely, and

$$\mathcal{N}\left(\left\{\left(y_k \eta_i^k\right)^{\zeta'}\right\}_{i,k}\right) =^{\mathcal{L}} \text{PD}\left(\frac{\zeta}{\zeta'}\right).$$

In particular, the law of the right-hand side does not depend on the sequence $\mathbf{y}$.

Proof. a) $\{(\eta_i, Y_i)\}_i$ is a Poisson point process on $\mathbb{R}^+ \times \mathbb{R}$ with intensity measure $\kappa(dx, dy) := \zeta x^{-\zeta-1}dx \otimes \mu(dy)$. If we map the points by the mapping $\phi : (x, y) \to \left(x\mathrm{e}^{\psi(y)}, y\right)$ we get a new Poisson point process, and we just have to compute the transformation of the intensity measure $\zeta x^{-\zeta-1}dx$ under ϕ: $\kappa\phi^{-1}(dx, dy) = \zeta x^{-\zeta-1}dx \otimes \mathrm{e}^{\zeta\psi(y)}\mu(dy)$.

Remark that this is again a marked Poisson point process with independent marks. Indeed, we can write

$$\zeta x^{-\zeta-1}dx\mathrm{e}^{\zeta\psi(y)}\mu(dy) = C(\zeta)\,\zeta x^{-\zeta-1}dx \cdot \mu'(dy),$$

where

$$C(\zeta) := \int \mathrm{e}^{\zeta\psi(y)}\mu(dy),$$
$$\mu'(dy) := \frac{1}{C(\zeta)}\mathrm{e}^{\zeta\psi(y)}\mu(dy).$$

So, $\left\{\left(\eta_i \mathrm{e}^{\psi(Y_i)}, Y_i\right)\right\}$ is equal in distribution with $\{(\eta_i', Y_i')\}$ with $\{\eta_i'\}$ being a PPP $\left(C(\zeta)\,\zeta x^{-\zeta-1}dx\right)$ with independent marks Y_i' having distribution μ'. However, the above PPP $\left(C(\zeta)\,\zeta x^{-\zeta-1}dx\right)$ is simply a PPP $\left(\zeta x^{-\zeta-1}dx\right)$ with the points stretched by a fixed factor $C(\zeta)^{1/\zeta}$.

b) follows from the basic transformation formula for Poisson point processes.

c) We compute the Laplace functional. Let $\phi \in C_0^+(\mathbb{R}^+)$. Then

$$\begin{aligned}
&E\left(\exp\left[-\sum\nolimits_{i,k}\phi\left(C(\zeta)^{-1/\zeta}\, y_k\eta_i^k\right)\right]\right)\\
&=\prod\nolimits_k E_k \exp\left[-\sum\nolimits_i \phi\left(C(\zeta)^{-1/\zeta}\, y_k\eta_i^k\right)\right]\\
&=\prod\nolimits_k \exp\left[-\int\left(1-\mathrm{e}^{\phi\left(C(\zeta)^{-1/\zeta}y_k x\right)}\right)\zeta x^{-\zeta-1}dx\right]\\
&=\prod\nolimits_k \exp\left[-\frac{y_k^\zeta}{C(\zeta)}\int\left(1-\mathrm{e}^{\phi(z)}\right)\zeta z^{-\zeta-1}dz\right]\\
&=\exp\left[-\sum_k\frac{y_k^\zeta}{C(\zeta)}\int\left(1-\mathrm{e}^{\phi(z)}\right)\zeta z^{-\zeta-1}dz\right]\\
&=\exp\left[-\int\left(1-\mathrm{e}^{\phi(z)}\right)\zeta z^{-\zeta-1}dz\right],
\end{aligned}$$

as claimed.

d) follows from b) and c). ■

Part a) of the proposition has the following consequence:

Corollary 4.2
Consider a PD (ζ) $\{\bar\eta_i\}$ *and independent* Y_i *with* $E\mathrm{e}^{\zeta\psi(Y_i)}<\infty$. *Define*

$$\hat\eta_i := \frac{\bar\eta_i \mathrm{e}^{\psi(Y_i)}}{\sum_j \bar\eta_j \mathrm{e}^{\psi(Y_j)}}$$

Then $\{(\hat\eta_i, Y_i)\}$ *is (as a point process) identical in law to* $\{(\bar\eta_i, Y_i')\}$ *with* Y_i' *independent marks with the law* μ' *as defined above.*

Proof. We represent $\{\bar\eta_i\}$ as $\mathcal{N}(\{\eta_i\})$, where $\{\eta_i\}$ is a PPP $\left(\zeta x^{-\zeta-1}dx\right)$, and apply the Proposition. As the scaling factor $C(\zeta)^{1/\zeta}$ cancels out after normalization, the claim follows. ■

We will draw an important conclusion from this corollary.

A PD (ζ) $\{\bar\eta_i\}$ can be interpreted as a random probability distribution on the integers $\mathbb{N}$. For that we have to use an ordering which is usually done by order them downwards: $\bar\eta_1 > \bar\eta_2 > \cdots$. Then 1 gets probability $\bar\eta_1$, 2 gets probability $\bar\eta_2$ etc. Actually all the final properties we derive don't depend on this ordering. We could start with the fifth largest, and go on in some way.

We denote this random probability law by $\mathcal{G}$, as it will turn out to be related to our random Gibbs measures. We can then also consider the product measures $\mathcal{G}^{(n)}$ on $\mathbb{N}^n$. One should keep in mind that these laws are random through the random character of the point process. We will write as usual $\mathbb{E}$ for the expectation under this, and use again $\nu^{(n)}$ for the probability measure $\int \mathcal{G}^{(n)}d\mathbb{P}$ on $\mathbb{N}^n$ but often drop the index n.

For $1\le i,j\le N$ we define the "overlap" $R_{ij}:\mathbb{N}^n\to\{0,1\}$ as $R_{ij}(\mathbf{x}) := \delta_{x_i,x_j}$, $\mathbf{x}\in\mathbb{N}^n$, and the matrix $R^{(n)} = (R_{ij})_{1\le i,j\le n}$. The aim now is to derive the Ghirlanda-Guerra-identity for the $R^{(n)}$:

Proposition 4.3

a)

$$\nu\left(R_{12}=1\right)=1-\zeta$$

b) For $n\geq 2$, the conditional law of $R_{1,n+1}$ given $R^{(n)}$ is

$$\frac{1}{n}\mathcal{L}_{\nu}\left(R_{12}\right)+\frac{1}{n}\sum_{k=2}^{n}\delta_{R_{1,k}.}$$

Firts proof. We apply Corollary 4.2 with $Y_i=g_i-\zeta$, g_i standard normal, and $\psi\left(y\right)=y$. In this case, μ' is just the standard normal distribution.

Let F be a real valued function defined on the set of $n\times n$-matrices, and consider the point process $\{(\hat{\eta}_i,Y_i)\}$ as defined above. Below, we compute expectations $\mathbb{E}$ with respect to this point process, and one has to take into account that the Y_i enter into the point process $\{\hat{\eta}_i\}$. However, $\{\hat{\eta}_i\}$ is simply a PD$\left(\zeta\right)$. From the corollary, we get

$$\begin{aligned}&\mathbb{E}\sum_{\mathbf{x}\in\mathbb{N}^n}\left(\hat{\eta}_{x_1}\cdot\cdots\cdot\hat{\eta}_{x_n}\right)F\left(R^{(n)}\left(\mathbf{x}\right)\right)Y_{x_1}\\&=\mathbb{E}\sum_{\mathbf{x}\in\mathbb{N}^n}\left(\bar{\eta}_{x_1}\cdot\cdots\cdot\bar{\eta}_{x_n}\right)F\left(R^{(n)}\left(\mathbf{x}\right)\right)Y'_{x_1}.\end{aligned}$$

In the point process $\{(\bar{\eta}_i,Y'_i)\}_i$, the Y'_i are independent marks. So the RHS above is 0 as the Y'_i are standard normal.

$$\hat{\eta}_i=\frac{\bar{\eta}_i\mathrm{e}^{g_i-\zeta}}{\sum_j\bar{\eta}_j\mathrm{e}^{g_j-\zeta}}.$$

Using the partial integration (Proposition 1.3), we get with $\nu^{(n)}:=\int\mathcal{G}^{(n)}d\mathbb{P}$

$$\begin{aligned}0&=\nu^{(n)}\left(F\left(R^{(n)}\left(\mathbf{x}\right)\right)\left(g_{x_1}-\zeta\right)\right)\\&=\sum_{k=1}^{n}\nu^{(n)}\operatorname{cov}\left(g_{x_1},g_{x_k}\right)F\left(R^{(n)}\left(\mathbf{x}\right)\right)\\&-n\nu^{(n+1)}\operatorname{cov}\left(g_{x_{n+1}},g_{x_1}\right)F\left(R^{(n)}\left(\mathbf{x}\right)\right)-\zeta\nu^{(n)}\left[F\left(R^{(n)}\left(\mathbf{x}\right)\right)\right].\end{aligned}$$

Here $\mathbf{x}$ is regarded as a random variable under ν. $\mathbb{E}$ finally integrates out the whole point process $\{(\hat{\eta}_i,Y_i)\}$ including the Gauss variables. Remark that $\operatorname{cov}\left(g_{x_1},g_{x_k}\right)=\delta_{x_1x_k}=R_{1,k}\left(\mathbf{x}\right)$. Taking into account $R\left(1,1\right)=1$, we obtain

$$\nu^{(n+1)}\left[R_{1,n+1}F\left(R^{(n)}\right)\right]=\frac{1-\zeta}{n}\nu^{(n)}F\left(R^{(n)}\right)+\frac{1}{n}\sum_{k=2}^{n}\nu^{(n)}\left[R_{1,k}F\left(R^{(m)}\right)\right].\tag{4.1}$$

In particular, with $n=1$, and $F=1$, we get

$$\nu^{(2)}\left(R_{1,2}\right)=\mathbb{E}\sum\bar{\eta}_i^2=1-\zeta,\tag{4.2}$$

which is a). Applying the above formula (4.1) for $n \geq 2$ then proves b). ■

Second proof. The above proof is from Panchenko's book [38]. A less elegant but maybe more intuitive argument runs as follows. I use a slightly non-rigorous formulation which can easily be made precise.

A point process $\{\eta_i\}$ which is a PPP $(f(t)\,dt)$ on $\mathbb{R}^+$, and for which $Z := \sum_i \xi_i < \infty$ almost surely, can be described as follows: In each infinitesimal interval $[t, t+dt]$ there is the chance $f(t)\,dt$ to have a point in this interval, and the different infinitesimal intervals are independent. Therefore, conditioned on $[t, t+dt]$ containing a point, the conditional distribution of Z is the unconditional distribution of $t+Z$. Therefore, for any $m > 0$, one has

$$\mathbb{E}\sum \bar{\eta}_i^m = \int_0^\infty dt\ f(t)\, t^m \mathbb{E}\left(\frac{1}{(t+Z)^m}\right). \tag{4.3}$$

Applying that to $f(t) = \zeta t^{-\zeta-1}$ and $m = 2$, we get

$$\begin{aligned}\nu(R_{1,2}) &= \int_0^\infty dt\ \zeta t^{-\zeta-1} t^2 \mathbb{E}\left(\frac{1}{(t+Z)^2}\right)\\ &= (1-\zeta)\int_0^\infty dt\ \zeta t^{-\zeta-1} t \mathbb{E}\left(\frac{1}{t+Z}\right)\\ &= (1-\zeta)\,\mathbb{E}\sum \bar{\eta}_i = (1-\zeta),\end{aligned}$$

by partial integration, which is a). b) can be proved along the same line.[6] ■

Exercise 4.4
Let again $\{\bar{\eta}_i\}$ be a $\mathrm{PD}(\zeta)$, and define for $\mathbf{r} = (r_1, \ldots, r_n) \in \mathbb{N}_0^n$ ($\mathbb{N}_0 := \{0, 1, 2, \ldots\}$)

$$X_{\mathbf{r}}^{(n)} := \sum\nolimits_{\mathbf{i}}^{*} \bar{\eta}_{i_1}^{r_1} \bar{\eta}_{i_2}^{r_2} \cdot \cdots \cdot \bar{\eta}_{i_n}^{r_n}, \tag{4.4}$$

where $\sum_{\mathbf{i}}^*$ means that we take the sum over n-tuples $\mathbf{i} = (i_1, \ldots, i_n)$ which are all distinct. Prove that (4.1) and (4.2) imply the following formula

$$\mathbb{E}X_{\mathbf{r}}^{(n)} = \frac{(n-1)!}{(N-1)!}\zeta^{n-1}\prod_{i=1}^{n} g(r_i, \zeta),$$

where $N := \sum_{k=1}^n r_k$ and

$$g(r,\zeta) := \begin{cases} 1 & \text{if } r = 1\\ (r-1-\zeta)(r-2-\zeta)\cdot\cdots\cdot(1-\zeta) & \text{if } r \geq 2 \end{cases}. \tag{4.5}$$

Exercise 4.5
Prove that (4.1) and (4.2) characterize $\mathrm{PD}(\zeta)$, that is if a point process $\{\xi_i\}$ on $\mathbb{R}^+$ satisfies $\sum \xi_i = 1$ and these two equations, then it has to be a $\mathrm{PD}(\zeta)$. This follows by the fact that the moments characterize the point process.

[6] Of course, the argument with infinitesimal intervals has to be replaced by a more careful reasoning to justify (4.3).

4.2 The GREM and the Ruelle cascades

Derrida evidently felt that the REM is too simple to shed any light on "real" spin glasses. He therefore invented a modification, called the "generalized random energy model", GREM for short, where the energies are correlated, like in the SK-model, but in a very special hierarchical way.

I will not prove anything of substance about Derrida's GREM, but I will quickly go to the limiting object, the Ruelle cascades.[7] I will however give a short description of the GREM, and discuss some of its properties, as it sheds some light on the concepts of the physicists, particularly what they understand by the notion of "pure states".

Consider a tree with a root and K levels. On each level, a bond branches into $2^{N/K}$ "children" bonds. The leaves which we call σ can then be written as

$$\sigma = (i_1, i_2, \dots, i_K)\,,\ 1 \leq i_j \leq 2^{N/K}.$$

(We assume that N is divisible by K). The bonds from $(i_1, \dots, i_{j-1})$ to $(i_1, \dots, i_j)$ of the graph can be identified with $(i_1, \dots, i_j)$, $j \leq K$. To pass from the root to the leaf σ on passes through the bonds

$$i_1, (i_1, i_2)\,, (i_1, i_2, i_3)\,, \dots, (i_1, i_2, \dots, i_K)\,.$$

The energies of the GREM are given by summing independent bond energies along the path from the root to the leaves.

$$H(\sigma) \stackrel{\text{def}}{=} X^{(1)}_{i_1} + X^{(2)}_{i_1,i_2} + \cdots + X^{(K)}_{i_1,\dots,i_K}.$$

All the $X^{(j)}$-variables are assumed to be independent and centered Gaussians. On level $j \leq K$, all variables have the same variances

$$\operatorname{var}\left(X^{(j)}_{i_1,\dots,i_j}\right) = \kappa_j^2 N.$$

Usually, one assumes that $\kappa_1^2 > \kappa_2^2 > \cdots > \kappa_K^2$, but it is not really necessary. (If it is not satisfied, then some of the levels simply disappear in the limit. See the discussion in Section 7.1). We also assume

$$\sum_{i=1}^{K} \kappa_i^2 = 1$$

which is a normalization of no importance.

The covariances are trivially computed

$$\mathbb{E}\left(H(\sigma)\, H\left(\sigma'\right)\right) = N \sum_{i=1}^{q(\sigma,\sigma')} \kappa_i^2,$$

[7] Ruelle did not prove that Derrida's GREM converges to the object he introduced, although he seemed to have taken it as a kind of "evident".

where

$$q\left((i_1,\dots,i_K),\left(i'_1,\dots,i'_K\right)\right)=\max\left(m:(i_1,\dots,i_m)=\left(i'_1,\dots,i'_m\right)\right).$$

Particularly, the variance of the variables is N as in the REM case.

The partition function and the Gibbs measure are defined in the usual way:

$$Z_{N,\beta,\omega}:=\sum_{\sigma}\exp\left[\beta H\left(\sigma\right)\right],\ \mathcal{G}_{N,\beta,\omega}\left(\sigma\right):=\frac{1}{Z_{N,\beta,\omega}}\exp\left[\beta H\left(\sigma\right)\right].$$

The free energy

$$f\left(\beta\right):=\lim_{N\to\infty}\frac{1}{N}\log Z_N$$

can be computed explicitly. It is piecewise quadratic with K pieces of different second derivative (provided $\kappa_1^2>\kappa_2^2>\cdots>\kappa_K^2$). The model has K critical values:

$$\beta_1^{\mathrm{cr}}:=\frac{\sqrt{2\log 2}}{\sqrt{K}\kappa_1}<\beta_2^{\mathrm{cr}}:=\frac{\sqrt{2\log 2}}{\sqrt{K}\kappa_2}<\cdots<\beta_K^{\mathrm{cr}}:=\frac{\sqrt{2\log 2}}{\sqrt{K}\kappa_K}.$$

For $\beta<\beta_1^{\mathrm{cr}}$, the free energy equals the annealed free energy. $f\left(\beta\right)=\beta^2/2+\log 2$. For $\beta>\beta_K^{\mathrm{cr}}$, the free energy is linear in β. The second derivative of the free energy in β jumps at all critical values down, but the first derivative stays continuous.

It may be instructive to discuss quickly $K=2$ which has all the ingredients.

Let's first look back at the random energy model which is the special case $K=1$, $\kappa_1^2=1$. There, for $s\geq 0$,

$$\#\left\{\sigma:H\left(\sigma\right)\approx sN\right\}\approx 2^N\exp\left[-\frac{s^2N}{2}\right],$$

with high probability. We are neglecting parts which are not exponentially in N. Also $\approx sN$ should mean $sN-o\left(N\right)\leq H\left(\sigma\right)\leq sN+o\left(N\right)$. If $s>\sqrt{2\log 2}$, then the right hand side goes to 0 exponentially fast, meaning that $\left\{\sigma:H\left(\sigma\right)\approx sN\right\}=\emptyset$ with high probability. In fact $\max_\sigma H\left(\sigma\right)$ is $\sqrt{2\log 2}N+o\left(N\right)$ with high probability, where $o\left(N\right)$ actually is only of order $\log N$. The free energy is then obtained through

$$\begin{aligned}Z_N&\approx\sum_{s\leq\sqrt{2\log 2}}\#\left\{\sigma:H\left(\sigma\right)\approx sN\right\}\mathrm{e}^{\beta sN}\\&\approx\exp\left[N\sup_{0\leq s\leq\sqrt{2\log 2}}\left(\beta s-s^2/2\right)\right]\end{aligned}$$

leading to the expression we have obtained in Section 2.5. For $\beta<\sqrt{2\log 2}$, the sup over s is attained at $s\left(\beta\right):=\beta$, and for $\beta\geq\sqrt{2\log 2}$, it is attained at $s\left(\beta\right):=\sqrt{2\log 2}$. This suggests (it's actually just an exercise) that the Gibbs distribution is concentrated on σ's for which $H\left(\sigma\right)\approx s\left(\beta\right)N$. The crucial distinction is whether $\beta<\sqrt{2\log 2}$ or not. In the former case, no single spin gets a Gibbs weight which is of order 1, and

in fact $\max \mathcal{G}(\sigma)$ is exponentially decaying. There are simply "too many" σ's which satisfy $\mathcal{G}(\sigma) \approx s(\beta) N$. On the other hand, if $\beta > \sqrt{2 \log 2}$, then the Gibbs weights just concentrate on the σ's for which $H(\sigma)$ is among the top values, and in the limit as $N \to \infty$, the Gibbs distribution is given by a Poisson-Dirichlet point process. In physics jargon, for $\beta < \sqrt{2 \log 2}$, there is just one "pure state", and for $\beta > \sqrt{2 \log 2}$, there are countably many with random weights, given by a PD. We abstain from discussing the border line case $\beta = \sqrt{2 \log 2}$ which is slightly tricky.

Let's now look at the modification for $K = 2$. We again want to compute the number of configurations σ with $H(\sigma) \approx sN$. This is now slightly more difficult. We best fix $s_1 < s$, and ask about about $\#A(s_1, s)$ where

$$A(s_1, s) := \left\{\sigma = (i_1, i_2) : X^1_{i_1} \approx s_1 N,\ X^2_{i_1, i_2} \approx (s - s_1) N\right\}$$

We first observe that with $A_1(s_1) := \left\{i_1 : X^1_{i_1} \approx s_1 N\right\}$ we have, as in the REM case

$$\#A_1(s_1) \approx 2^{N/2} \exp\left[-\frac{s_1^2 N}{2\kappa_1^2}\right] = \exp\left[N\left\{\frac{1}{2}\log 2 - \frac{s_1^2}{2\kappa_1^2}\right\}\right].$$

The exponent gets negative for

$$s_1 > \kappa_1 \sqrt{\log 2}$$

which means that $A_1(s_1) = \emptyset$ and therefore also $A(s_1, s) = \emptyset$ with high probability. So, we have to restrict to $s_1 \leq \kappa_1 \sqrt{\log 2}$. In that case

$$\begin{aligned}
\#A(s_1, s) &\approx \#A_1(s_1) \times \#\left\{i_2 : X^2_{1, i_2} \approx (s - s_1) N\right\} \\
&\approx \exp\left[N\left\{\frac{1}{2}\log 2 - \frac{s_1^2}{2\kappa_1^2}\right\}\right] \exp\left[N\left\{\frac{1}{2}\log 2 - \frac{(s - s_1)^2}{2\kappa_2^2}\right\}\right] \\
&= \exp\left[N\left\{\log 2 - \frac{s_1^2}{2\kappa_1^2} - \frac{(s - s_1)^2}{2\kappa_2^2}\right\}\right].
\end{aligned}$$

It is not difficult to prove that, up to corrections of subexponential order, one has

$$\begin{aligned}
\#A(s) &= \sup_{s_1 : s_1 \leq \kappa_1 \sqrt{\log 2}} \exp\left[N\left\{\log 2 - \frac{s_1^2}{2\kappa_1^2} - \frac{(s - s_1)^2}{2\kappa_2^2}\right\}\right] \\
&= \exp\left[N\left\{\log 2 - \inf_{s_1 : s_1 \leq \kappa_1 \sqrt{\log 2}} \left(\frac{s_1^2}{2\kappa_1^2} + \frac{(s - s_1)^2}{2\kappa_2^2}\right)\right\}\right].
\end{aligned}$$

Write

$$\phi(s) := \inf_{s_1 : s_1 \leq \kappa_1 \sqrt{\log 2}} \left(\frac{s_1^2}{2\kappa_1^2} + \frac{(s - s_1)^2}{2\kappa_2^2}\right) \tag{4.6}$$

For fixed s the unrestricted minimum in s_1 is attained at

$$s_1 = \kappa_1^2 s.$$

This is however only $\leq \kappa_1\sqrt{\log 2}$ if

$$s \leq \frac{\sqrt{\log 2}}{\kappa_1} < \sqrt{2\log 2}$$

as we had assumed $\kappa_1^2 + \kappa_2^2 = 1$ and $\kappa_1^2 > \kappa_2^2$. For s satisfying this restriction, we have

$$\phi(s) = \frac{s^2}{2}$$

which is the same as for the REM. On the other hand, if $s > \frac{\sqrt{\log 2}}{\kappa_1}$ we have

$$\phi(s) = \frac{\log 2}{2} + \frac{\left(s - \kappa_1\sqrt{\log 2}\right)^2}{2\kappa_2^2}.$$

Therefore, $\#A(s)$ can be $\neq 0$ only if $\phi(s) \leq \log 2$, i.e.

$$s \leq (\kappa_1 + \kappa_2)\sqrt{\log 2}.$$

Having computed $\phi(s)$ for all s, it is easy to see that

$$f(\beta) = \lim_{N\to\infty} \frac{1}{N} \log Z_{N,\beta} = \sup_{s \leq (\kappa_1+\kappa_2)\sqrt{\log 2}} (\beta s - \phi(s)) + \log 2$$

The reader may easily check that for $\beta \leq \beta_1^{\mathrm{cr}} = \kappa_1^{-1}\sqrt{\log 2}$, the supremum is attained at $s = s(\beta) := \beta$ leading to $f(\beta) = \beta^2/2 + \log 2$. For $\beta_1^{\mathrm{cr}} = \kappa_1^{-1}\sqrt{\log 2} \leq \beta \leq \beta_2^{\mathrm{cr}} = \kappa_2^{-1}\sqrt{\log 2}$, the supremum is attained at

$$s(\beta) := \kappa_1\sqrt{\log 2} + \kappa_2^2\beta \in \left[\frac{\sqrt{\log 2}}{\kappa_1}, (\kappa_1 + \kappa_2)\sqrt{\log 2}\right].$$

Finally, for $\beta > \beta_2^{\mathrm{cr}}$, the supremum is attained at $s(\beta) := (\kappa_1 + \kappa_2)\sqrt{\log 2}$. Remark that $(\kappa_1 + \kappa_2)\sqrt{\log 2}N$ is the maximum of $H(\sigma)$ (up to smaller order in N). We write also $s_1(\beta)$ for the maximizer in the variational formula (4.6) which satisfies $s_1(\beta) < \kappa_1\sqrt{\log 2}$ if and only if $\beta < \beta_1^{\mathrm{cr}}$. (Remember that $\kappa_1\sqrt{\log 2}N$ was in leading order the maximal value of $X_i^{(1)}$, $1 \leq i \leq 2^{N/2}$. Plugging that in, one gets the explicit formula for the free energy which is not very interesting. More interesting is what the above analysis suggests for the behavior of the Gibbs distribution. (For more details about that, see [11] and [18]).

For $\beta < \beta_1^{\mathrm{cr}}$, nothing interesting happens: The Gibbs measure concentrates at σ's for which $H(\sigma) \approx s(\beta)N$ which means that it concentrates on $\sigma = (i_1, i_2)$ where $X_{i_1}^{(1)} \approx s_1(\beta)N$, $X_{i_1 i_2}^{(2)} \approx (s(\beta) - s_1(\beta))N$, up to smaller order in N, and the reader will have no difficulty to check that in fact the maximal Gibbs weight is exponential small in N. Not only that, also the marginal distribution on the first level

$$\mathcal{G}^1(i_1) = \sum_{i_2} \mathcal{G}((i_1, i_2))$$

has the property that $\max_{i_1} \mathcal{G}^1(i_1)$ is exponentially small in N.

In the region $\beta_1^{\rm cr} < \beta < \beta_2^{\rm cr}$. The Gibbs distribution concentrates on $\sigma = (i_1, i_2)$ where $X^{(1)}_{i_1} \approx \kappa_1 \sqrt{\log 2} N$ (up to smaller order in N), and $X^{(2)}_{i_1 i_2} \approx \left(s(\beta) - \kappa_1 \sqrt{\log 2}\right) N$. The maximum of the Gibbs distribution is still exponentially small in N, but now $\mathcal{G}^1$ concentrates on i_1 for which $X^{(1)}_{i_1}$ is maximal. This then implies that $\mathcal{G}^1$ remains macroscopic in the $N \to \infty$ limit, and in fact

$$\sum_{i_1} \delta_{\mathcal{G}^1(i_1)}$$

converges weakly to a Poisson-Dirichlet point process. One says that the Gibbs distribution "freezes" on the first level. In physics jargon, there are countably many pure states in this case.[8] They consist of the exponentially large collection of σ's which have i_1's for which $\mathcal{G}^1(i_1)$ belongs to the top ones. Finally for $\beta > \beta_2^{\rm cr}$ the Gibbs distribution itself freezes, and $\sum_\sigma \delta_{\mathcal{G}(\sigma)}$ converges to a Poisson-Dirichlet point process. In that case, the "pure states" are the single configurations.

The most interesting case is the intermediate one $\beta_1^{\rm cr} < \beta < \beta_2^{\rm cr}$. There, the individual configurations have only exponential small Gibbs weights. The configurations can however be bundled into lumps with the same first component i_1. The lumps then have Gibbs weights given by Poisson-Dirichlet, and they are what in physics literature are called "pure states". In physics literature on mean-field spin glasses like in [35], it is suggested that something like that should be true in other more interesting models like the SK-model. There is however no mathematical proof except in some special cases of p-spin models which are investigated in [42].

We don't give any more details about that, which has been thoroughly investigated by Bovier and Kurkova [18], but now present Ruelle's limit object [39].

Ruelle argued that the limit Gibbs measure (for β large) should have the following cascade structure. One chooses K parameters $0 < \zeta_1 < \cdots < \zeta_K < 1$. Then, on a first level one chooses a PPP $\left(\zeta_1 x^{-\zeta_1 - 1} dx\right)$, $\Xi^1 = \left\{\eta_i^1\right\}_i$. On the next level, one chooses for *any* $i \in \mathbb{N}$ a PPP $\left(\zeta_2 x^{-\zeta_2 - 1} dx\right)$ Ξ_i^2 whose countably many points are denoted by $\left\{\eta_{ij}^2\right\}_{j \in \mathbb{N}}$, and we furthermore assume that these point processes are all independent, and also independent of Ξ^1. In this way, one proceeds: On the third level, one chooses independent point processes $\Xi^3_{i_1 i_2} \stackrel{\rm def}{=} \left\{\eta^3_{i_1 i_2 j}\right\}_j$ for any $i_1, i_2 \in \mathbb{N}$, and these point processes have intensity measure $\zeta_3 x^{-\zeta_3 - 1} dx$.

Such a cascade of point processes $\Xi^1, \Xi^2_{i_1}, \Xi^3_{i_1 i_2}, \ldots, \Xi^K_{i_1 i_2 \ldots i_{K-1}}$ is called a **Ruelle cascade** to the parameter $(\zeta_1, \ldots, \zeta_K)$.

We can multiply the points of the all the processes: For $\mathbf{i} = (i_1, \ldots, i_K) \in \mathbb{N}^K$, we put

$$\eta_{\mathbf{i}} \stackrel{\rm def}{=} \eta^1_{i_1} \eta^2_{i_1 i_2} \cdots \eta^K_{i_1, i_2, \ldots, i_K} \tag{4.7}$$

[8] Strictly speaking, this does not make any mathematically precise sense as it would refer to a $N = \infty$ situation. The proper notion will appear later in Section 7.2

This leads to the point process

$$\Xi^{\mathrm{tot}} \stackrel{\mathrm{def}}{=} \{\eta_{\mathbf{i}}\}$$

One should think of the points of this point process as the unnormalized Gibbs weights of a limiting GREM-type spin glass. For a proof of this fact, see [18]. In a special case, it was also proved in [11].

The first surprise is

Proposition 4.6
$\mathcal{N}\left(\Xi^{\mathrm{tot}}\right)$ is stochastically independent of the cascade up to level $K-1$ and its law is $\mathrm{PD}\left(\zeta_K\right)$.

Proof. Take $K=2$. Remark that because of $\zeta_1<\zeta_2<1$ we have $\int_{(0,1)} x^{\zeta_2}\zeta_1 x^{-\zeta_1-1}dx$ $<\infty$, and therefore

$$C := \sum_i \left(\eta_i^1\right)^{\zeta_2} < \infty \tag{4.8}$$

almost surely. We can now apply Proposition 4.1. For that we condition on the first level $\left\{\eta_i^1\right\}$, and apply the Proposition with $y_i \stackrel{\mathrm{def}}{=} \eta_i^1$, $\zeta=\zeta_2,\ \zeta'=1$. Then

$$\mathcal{N}\left(\left\{\eta_i^1\eta_{ij}^2\right\}\right) =^{\mathcal{L}} \mathrm{PD}\left(\zeta_2\right).$$

That is the *conditioned* law on the first level. As this law does not depend on $\left\{\eta_i^1\right\}$, the statement follows.

The general K case follows easily by induction, always conditioning on the level $K-1$. ■

At first sight, this proposition seems to tell that the introduction of the cascade structure does not give anything new which is not already present in the case $K=1$, but this is wrong, as there is a non-trivial notion of an overlap structure.

To discuss it, it is best to order the points downwards. Therefore, we assume $\eta_1^1>\eta_2^1>\cdots$ and for any $i:\eta_{i1}^2>\eta_{i2}^2>\eta_{i3}^2>\cdots$ etc. In this way, the index set $\mathbf{i}$ for $\eta_{\mathbf{i}}$ is identified with $\mathbb{N}^K$. We can order the $\eta_{\mathbf{i}}$ downwards which leads to a random bijection $\phi:\mathbb{N}\to\mathbb{N}^K:\eta_{\phi(k)}$ is the k-th biggest among the $\eta_{\mathbf{i}}$.

Let $0\le k\le K$ and fix it for the moment. We define a (random) equivalence relation on $\mathbb{N}$ by setting

$$i\sim_k j \stackrel{\mathrm{def}}{\Longleftrightarrow} \phi\left(i\right)_r = \phi\left(j\right)_r \ \forall r\le k$$

In other words, i is equivalent to j if and only if the branching between the i-th largest and the j-th largest is at level k or later.

The equivalence relation induces a partition $\mathcal{Z}_k$ of $\mathbb{N}$ into disjoint subsets, the equivalence classes under the equivalence relations $\sim_k$. By the very definition, it is clear that $\mathcal{Z}_{k+1}$ is a finer partition than $\mathcal{Z}_k$. If $\mathcal{Z}$ and $\mathcal{Z}'$ are to partitionings of $\mathbb{N}$ we write $\mathcal{Z}'\prec\mathcal{Z}$ if $\mathcal{Z}$ is obtained by possibly dividing the sets of $\mathcal{Z}'$, i.e. if it is the finer partitioning. Using this notation, we evidently have

$$\mathcal{Z}_K = \left\{\{i\}:i\in\mathbb{N}\right\}\succ\mathcal{Z}_{K-1}\succ\cdots\succ\mathcal{Z}_1\succ\mathcal{Z}_0=\left\{\mathbb{N}\right\}. \tag{4.9}$$

The Ruelle cascade therefore leads to a sequence of random partitionings $(\mathcal{Z}_0, \mathcal{Z}_1, \ldots, \mathcal{Z}_K)$. Of course only the $\mathcal{Z}_k$ with $1 \leq k \leq K-1$ are random.

A most remarkable property is

Proposition 4.7
$\mathcal{N}(\{\eta_{\mathfrak{i}}\})$ and $(\mathcal{Z}_0, \mathcal{Z}_1, \ldots, \mathcal{Z}_K)$ are stochastically independent.

Proof. We do the proof only for $K = 2$ which contains essentially all the ingredients. The general case is only notational more cumbersome. For $K = 2$, the only point is to prove that $\mathcal{Z}_1$ is independent of $\mathcal{N}(\{\eta_{\mathfrak{i}}\})$.

We start with a sequence $\mathbf{y} = \{y_k\}$ of positive reals, and independent point processes $\left\{\eta^2_{ki}\right\}_i$ which are PPP $\left(\zeta_2 x^{-\zeta_2 - 1} dx\right)$. For notational simplicity, we drop the index 2 for the moment. Then consider $\mathcal{N}\left(\left\{y_k \eta_{ki}\right\}_{i,k}\right)$. Assuming $C(\zeta, \mathbf{y}) := \sum y_k^{\zeta} < \infty$, we know that it is a PD(ζ). We order the points $y_k \eta_{ki}$ downwards, creating in this way the random partitionings $\mathcal{Z}$ of the natural numbers. Formally, we simply attach to a point of the point-process $\mathcal{N}\left(\left\{y_k \eta_{ki}\right\}_{i,k}\right)$ the number ℓ if the point stems from the group $y_l \eta_{\ell \cdot}$. This creates a *marked* point process with marks in $\mathbb{N}$.

We now prove that this marked point process is a PD(ζ) point process with independently attached marks in $\mathbb{N}$, where the law of the marks depends on the sequence $\{y_k\}$. Having proved that, we have proved the proposition applying it to $\{y_k\} = \left\{\eta^1_k\right\}$ which is, as a point process, independent of $\mathcal{N}\left(\left\{\eta^1_k \eta^2_{ki}\right\}_{i,k}\right)$.

A point process with values in $\mathbb{R}^+$ with marks in $\mathbb{N}$ is a point process with values in $\mathbb{R}^+ \times \mathbb{N}$ which has the property that almost surely, one has for all $s \in \mathbb{R}^+$ there is at most one point in $\{s\} \times \mathbb{N}$. One also requires that the projection of the points to $\mathbb{R}^+$ gives a point process on $\mathbb{R}^+$ (which is not automatic from the requirement that one has a point process on $\mathbb{R}^+ \times \mathbb{N}$). On the other hand, there is no requirement that the projection onto $\mathbb{N}$ leads to a point process on $\mathbb{N}$. A very special case is the one where *independent* marks are attached to a point process on $\mathbb{R}^+$. We have encountered that before (with marks in $\mathbb{R}$).

We compute the Laplace functional of a point process $\{\eta_i\}$ with independent marks X_i in $\mathbb{N}$ which are distributed according to probability weights $p = \{p(k)\}_{k \in \mathbb{N}}$. If $\phi \in C_o^+(\mathbb{R} \times \mathbb{N})$, then

$$\begin{aligned} E \exp\left[-\sum\nolimits_i \phi(\eta_i, X_i)\right] &= \sum_k p(k)\, E \exp\left[-\sum\nolimits_i \phi(\eta_i, k)\right] \\ &= E \exp\left[-\sum\nolimits_i \psi(\eta_i)\right], \end{aligned}$$

where

$$\mathrm{e}^{-\psi(y)} = \sum_k p(k)\, \mathrm{e}^{-\phi(y,k)}.$$

So, if $\{\eta_i\}$ is a PPP $\left(\zeta x^{-\zeta-1}dx\right)$, this is

$$\begin{aligned}&\exp\left[-\int\left(1-\mathrm{e}^{-\psi(x)}\right)\zeta x^{-\zeta-1}dx\right]\\&=\exp\left[-\sum\nolimits_k p\left(k\right)\int\left(1-\mathrm{e}^{-\phi(x,k)}\right)\zeta x^{-\zeta-1}dx\right].\end{aligned}$$

We apply this to $\left\{\left(y_k\eta_{ki},k\right)_{i,k}\right\}$: Let $\phi:\mathbb{R}^+\times\mathbb{N}\to\mathbb{R}^+$ be continuous with compact support. Then

$$\begin{aligned}E\exp\left[-\sum\nolimits_{i,j}\phi\left(y_k\eta_{ki},k\right)\right]&=\prod\nolimits_k E\exp\left[-\sum\nolimits_i\phi\left(y_k\eta_{ki},k\right)\right]\\&=\prod\nolimits_k\exp\left[-\int\left(1-\mathrm{e}^{-\phi(y_kx,k)}\right)\zeta x^{-\zeta-1}dx\right]\\&=\prod\nolimits_k\exp\left[-\int\left(1-\mathrm{e}^{-\phi(x,k)}\right)\frac{\zeta}{y_k}\left(\frac{x}{y_k}\right)^{-\zeta-1}dx\right]\\&=\exp\left[-\sum\nolimits_k\int\left(1-\mathrm{e}^{-\phi(x,k)}\right)p_{\zeta,\mathbf{y}}\left(k\right)C\left(\zeta,\mathbf{y}\right)\zeta x^{-\zeta-1}dx\right],\end{aligned}$$

where

$$p_{\zeta,\mathbf{y}}\left(k\right):=y_k^{\zeta}/C\left(\zeta,\mathbf{y}\right).$$

This proves that, for fixed sequences $\mathbf{y}$ with $C\left(\zeta,\mathbf{y}\right)<\infty$, the point process

$$\left\{\left(y_k\eta_{ki},k\right)_{k,i}\right\}$$

is a marked point process which is a PPP $\left(C\left(\zeta,\mathbf{y}\right)\zeta x^{-\zeta-1}dx\right)$ with independently attached points in $\mathbb{N}$ with law $p_{\zeta,\mathbf{y}}$, and therefore $\left\{\left(y_k\eta_{ki}/C\left(\zeta,\mathbf{y}\right)^{1/\zeta},k\right)_{k,i}\right\}$ is a PPP $\left(\zeta x^{-\zeta-1}dx\right)$ with independently attached marks in $\mathbb{N}$. After normalization

$$\left\{\left(\frac{y_k\eta_{ki}}{\sum_{\ell,j}y_l\eta_{\ell j}},k\right)_{i,k}\right\}$$

we get a PD $\left(\zeta\right)$ with independent marks in $\mathbb{N}$, distributed according to $p_{\zeta,\mathbf{y}}$.

The statement (for $K=2$) now follows easily. Conditionally on the first level $\left\{\eta^1_{i_1}\right\}_{i_1}$, the clustering is through marks, independently attached to the point process $\Xi=\mathcal{N}\left(\left\{\eta^1_{i_1}\eta^2_{i_1i_2}\right\}\right)$. The law of the latter does not depend on the realization of $\left\{\eta^1_i\right\}$, however the distribution of the marks does. As $\left\{\eta^1_i\right\}$ and $\mathcal{N}\left(\left\{\eta^1_{i_1}\eta^2_{i_1i_2}\right\}\right)$ are independent by Proposition 4.6, the claim follows. ■

Therefore, the clustering is stochastically independent of Ξ^{tot}, and is obtained through a two-stage procedure: Depending on η^1, one computes a probability law on $\mathbb{N}$ through

$$\left\{p_{\zeta_2,\eta^1}\left(i\right)\right\}_{i\in\mathbb{N}}$$

and chosen conditionally independent marks, given η^1, the marks attached to the point process Ξ. The matching is performed by matching points with the same marks. The total distribution of the matching is obtained by choosing η^1 according to its law.

Remark 4.8
It is important to carefully spell out the mechanism of the matching procedure. Conditionally on the first level η^1, both, the point process Ξ, and the matching depend on the point process η^2, and are stochastically independent. The point process Ξ is however independent of η^1, whereas the matching is not. Nevertheless, this implies that Ξ and the matching are also unconditionally independent.

The point process $\left\{p_{\zeta_2,\eta^1}(i)\right\}_{i\in\mathbb{N}}$ itself is obtained through normalizing $\left\{\left(\eta_i^1\right)^{\zeta_2}\right\}$, i.e. it is a PD (ζ_2/ζ_1) according to Proposition 4.1 d).

Another important property of the Ruelle cascades is that they satisfy the Ghirlanda-Guerra identities. It is irrelevant if one formulates this in terms of the original indices for $\eta_{\mathbf{i}}^{\mathrm{tot}}$, or after reordering the weights in decreasing order. Let's do the former, as then the overlap is "not random". We define the the overlap $R\left(\mathbf{i},\mathbf{i}'\right)$ to be the largest number k such that $(i_1,\ldots,i_k)=(i'_1,\ldots,i'_k)$. Here, these are numbers in $\{0,1,\ldots,K\}$. For the formulation of the Ghirlanda-Guerra identity, we can take this as the overlap. Later we will use a monotone map $\{0,1,\ldots,K\}\to[0,1]$, but for the moment, this is irrelevant. We take the normalized weights $\bar{\eta}_{\mathbf{i}}^{\mathrm{tot}}$ which defines a random probability distribution in $\mathbb{N}^K$, and take product measures. We can as well take the infinite one, but it does not matter for the formulation. Integrating out, after taking the product measure: $\nu:=\int\left(\bar{\eta}^{\mathrm{tot}}\right)^{\otimes\mathbb{N}}d\mathbb{P}$, where $\mathbb{P}$ refers to the law for the Ruelle cascades, we arrive at a probability measure on $\left(\mathbb{N}^K\right)^{\mathbb{N}}$. We write $\mathbf{I}_k$ for the projections $\left(\mathbb{N}^K\right)^{\mathbb{N}}\to\mathbb{N}^K$.

Proposition 4.9
Let $n\in\mathbb{N}$. Under ν, the conditional distribution of $R(\mathbf{I}_1,\mathbf{I}_{n+1})$ given the random matrix $(R(\mathbf{I}_j,\mathbf{I}_k))_{j,k\leq n}$ is

$$\frac{1}{n}\sum_{j=2}^{n}\delta_{R(\mathbf{I}_1,\mathbf{I}_j)}+\frac{1}{n}\mathcal{L}_{R(\mathbf{I}_1,\mathbf{I}_2)}.$$

Proof. I don't give a proof. It is a good exercise to try to prove the $K=2$ case. ∎

The Ghirlanda-Guerra identity characterizes the Ruelle cascades in a way which will be made precise later.

4.3 The coalescent process

The aim of this section is to give further information about the structure of the distribution of $(\mathcal{Z}_0,\mathcal{Z}_1,\ldots,\mathcal{Z}_K)$. For that, we define a continuous time , time homogeneous, Markov process $\{\Gamma_t\}_{t\geq0}$ taking values in the set of partitionings of $\mathbb{N}$ (or equivalently, in the set of equivalence relations). We write $\Pi_{\mathbb{N}}$ for the set of equivalence relations on $\mathbb{N}$. The set of equivalence relations is a subset of the set of all relations on $\mathbb{N}$. The latter is evidently a compact set, as it can be presented as a subset of $\{0,1\}^{\mathcal{P}(\mathbb{N})}$, where $\mathcal{P}(\mathbb{N})$

denotes the set of (unordered) pairs of $\mathbb{N}$.. It is readily checked that $\Pi_{\mathbb{N}}$ is a closed subset of the set of relations, and therefore, it is compact as well.

If I is a finite subset of $\mathbb{N}$, then we write Π_I for the set of equivalence relations on I. This is a finite set. We write $\pi_{J,I}$ for the projection $\Pi_J \to \Pi_I$.

We construct the process $\{\Gamma_t\}$ via its projections $\Gamma_{t,I} \stackrel{\text{def}}{=} \pi_{\mathbb{N},I}(\Gamma_t)$ for $I \subset\subset \mathbb{N}$. $\{\Gamma_{t,I}\}$ itself is a Markov process which is not automatic from the Markov property of $\{\Gamma_t\}$, of course. Anyway, $\{\Gamma_{t,I}\}$ being a continuous time Markov process on the finite set Π_I, it is perfectly described by its transition matrix $R_{t,I}(\gamma, \gamma')$, $\gamma, \gamma' \in \Pi_I$, which then can be written as

$$R_{t,I} = \exp[tQ_I],$$

with the Q-matrix $Q_I(\gamma, \gamma')$, satisfying

$$\sum_{\gamma'} Q(\gamma, \gamma') = 0, \ \forall \gamma,$$
$$Q(\gamma, \gamma') \geq 0, \ \forall \gamma \neq \gamma'.$$

Here it is: Transitions are possible only to coarser partitionings, i.e. from γ to a $\gamma' \prec \gamma$. Therefore, if γ has just one class, then no transitions are possible, and this is absorbing. This means that $Q_I(\gamma, \gamma') = 0$ if $|\gamma| = 1$. $|\gamma|$ here the number of classes in γ. If $|\gamma| = N \geq 2$, and if γ' is obtained from γ by clumping together exactly $k \geq 2$ classes, then

$$Q_I(\gamma, \gamma') \stackrel{\text{def}}{=} \frac{1}{(N-1)\binom{N-2}{k-2}}.$$

All other $Q_I(\gamma, \gamma')$ with $\gamma' \neq \gamma$ are 0. So, infinitesimally, only one clumping act is possible, but the number of clumped sets is not restricted. Furthermore, of course,

$$Q_I(\gamma, \gamma) = -\sum_{\gamma' : \gamma' \neq \gamma} Q_I(\gamma, \gamma').$$

This defines in the standard way a Markov process $\{\Gamma_{t,I}\}_{t \geq 0}$.

Exercise 4.10
The transitions of the Markov process $\{\Gamma_{t,I}\}$, $I \subset\subset \mathbb{N}$, are given in the following way. Conditioned on $\{\Gamma_{t,I} = \gamma\}$, the process stays in γ for an exponential time with expectation $(|\gamma| - 1)^{-1}$. (Of course, if $|\gamma| = 1$, then the process stays there forever). At the jump time, $\xi \in \{2, \ldots, |\gamma|\}$ classes are clumped with

$$P(\xi = k) = \frac{|\gamma|}{|\gamma| - 1} \frac{1}{k(k-1)}.$$

Conditioned on $\{\xi = k\}$, the k classes to be clumped together in one new class are chosen with equal probability among the $\binom{N}{k}$ possibilities.

With probability one, the process reaches the absorbing one-class state after a finite time.

Somewhat surprisingly, the transition kernel $R_{t,I}$ for I finite, can be computed explicitly:

Proposition 4.11
Assume $\gamma \in \Pi_I$ has N classes, and γ' is obtained by clumping $r_1, r_2, \ldots, r_k \geq 1$ classes of γ, with $\sum_i r_i = N$. Then

$$R_{t,I}\left(\gamma, \gamma'\right) = \frac{(k-1)!}{(N-1)!} \mathrm{e}^{-(k-1)t} \prod_{i=1}^{k} g\left(r_j, \mathrm{e}^{-t}\right), \tag{4.10}$$

where $g(r, \zeta)$ is defined in (4.5).

Proof. We write $q_t\left(\gamma, \gamma'\right)$ for the right-hand side of (4.10). Evidently, q_0 is the identity matrix. We prove

$$\frac{dq_t}{dt} = Q q_t.$$

From that, the claim follows.

We write $x = \mathrm{e}^{-t}$, $f_x(s) = s^x$, $s > 0$. Then

$$q_t\left(\gamma, \gamma'\right) = (-1)^{N-k} \frac{(k-1)!}{(N-1)!} \prod_{i=1}^{k} f_x^{(r_i)}(1),$$

where $f^{(m)}$ denotes the m-th derivative w.r.t. s.

For $m \geq 1$, one has

$$\begin{aligned} \frac{\partial f_x^{(m)}(1)}{\partial t} &= \frac{\partial^m}{\partial s^m} \left(-x \log s\right) f_x(s) \Big|_{s=1} \\ &= x \sum_{j=1}^{m} (-1)^j \binom{m}{j} (j-1)! f_x^{(m-j)}(1). \end{aligned}$$

The functions $f_x(s)$ satisfy the identity

$$x f_x^{(r)}(1) = f_x^{(r+1)}(1) + r f_x^{(r)}(1),$$

and implementing that, we get

$$\frac{\partial f_x^{(m)}(1)}{\partial t} = m! \sum_{j=2}^{m} (-1)^{j-1} \frac{f_x^{(m-j+1)}(1)}{j(j-1)(m-j)!} - m f_x^{(m)}(1),$$

where the sum over j is 0 in case $m = 1$.

Let now γ, γ' be as in the statement of the proposition. Then

$$\begin{aligned}
\frac{dq_t(\gamma,\gamma')}{dt} &= (-N+1)\, q_t(\gamma,\gamma') \\
&\quad + (-1)^{N-k}\, \frac{(k-1)!}{(N-1)!} \\
&\quad \times \sum_{i:m_i\geq 2} \sum_{r=2}^{m_i} (-1)^{r-1}\, \frac{m_i! f_x^{(m_i-r+1)}(1)}{(m_i-r)! r\,(r-1)\, x} \prod_{j:j\neq i} f_x^{(m_j)}(1) \\
&= (-N+1)\, q_t(\gamma,\gamma') \\
&\quad + \sum_{i:m_i\geq 2} \sum_{r=2}^{m_i} \frac{1}{(N-1)\binom{N-2}{r-2}} \binom{m_i}{r} (-1)^{N-k-r+1} \\
&\quad \times \frac{(k-1)!}{(N-r)!} \frac{1}{x} f_x^{(m_i-r+1)}(1) \prod_{j:j\neq i} f_x^{(m_j)}(1) \\
&= \sum_{\gamma'':\gamma'\prec\gamma''\prec\gamma} Q(\gamma,\gamma'')\, q_t(\gamma'',\gamma').
\end{aligned}$$

This proves the claim. ■

We next claim that the Markov processes on $\Pi_I,\ I \subset\subset \mathbb{N}$, are compatible, meaning that if $I \subset J$, then the Markov process constructed with values in Π_J, projected onto Π_I is the Markov process with this state space. This is proved by checking that the Q-matrices have the appropriate compatibility property, namely

Lemma 4.12
Let $\gamma, \gamma' \in \Pi_I,\ \gamma' \prec \gamma$, and $\tilde{\gamma}$ be any element in Π_J with $\pi_{J,I}(\tilde{\gamma}) = \gamma$. Then

$$Q_I(\gamma,\gamma') = \sum_{\substack{\tilde{\gamma}'\in\Pi_J:\tilde{\gamma}'\prec\tilde{\gamma},\\ \pi_{J,I}(\tilde{\gamma}')=\gamma'}} Q_J(\tilde{\gamma},\tilde{\gamma}').$$

Proof. We only have to check the formula when γ' is obtained from γ by clumping k classes, γ having $N \geq k$ classes, $k \geq 2$. The chosen $\tilde{\gamma}$ may have N classes, too, or more. Say, it has $\tilde{N} \geq N$ classes. Now, in order to get by a simple clumping a partitioning $\tilde{\gamma}'$ which when restricted to I equals γ', one has several possibilities, but certainly, the extensions of the classes clumped in γ have to be clumped. Of the $\tilde{N} - N$ new classes in $\tilde{\gamma}$, the clumping of them has no influence on the trace on I. Therefore, if we decide to clump $l \leq \tilde{N} - N$ of them, there are simply $\binom{\tilde{N}-N}{l}$ to select this group which should be clumped, and in this case, we have

$$Q_J(\tilde{\gamma},\tilde{\gamma}') = \frac{1}{\left(\tilde{N}-1\right)\binom{\tilde{N}-2}{k+l-2}}.$$

Therefore, all we have to check is

$$\frac{1}{(N-1)\binom{N-2}{k-2}} = \sum_{l=0}^{\tilde{N}-N} \binom{\tilde{N}-N}{l} \frac{1}{\left(\tilde{N}-1\right)\binom{\tilde{N}-2}{k+l-2}}$$

which is elementary. ■

As a consequence, one obtains the compatibility of the semigroups $R_{t,I}$, namely that if $I \subset J$, and $\tilde{\gamma} \in \Pi_J$, then

$$R_{t,I}\left(\pi_{J,I}\left(\tilde{\gamma}\right), \cdot\right) = R_{t,J}\left(\tilde{\gamma}, \pi_{J,I}^{-1}\left(\cdot\right)\right). \tag{4.11}$$

and then, by soft arguments, one can extend the semigroup to a Feller semigroup $\{R_t\}$ on $\Pi_{\mathbb{N}}$, satisfying

$$R_{t,I}\left(\pi_{\mathbb{N},I}\left(\tilde{\gamma}\right), \cdot\right) = R_t\left(\tilde{\gamma}, \pi_{\mathbb{N},I}^{-1}\left(\cdot\right)\right) \tag{4.12}$$

for any $I \subset\subset \mathbb{N}$, $\tilde{\gamma} \in \Pi_{\mathbb{N}}$. This leads to a Feller process $\{\Gamma_t\}_{t\geq 0}$ taking values in $\Pi_{\mathbb{N}}$ which we start with the trivial partitioning of $\mathbb{N}$ into single points. This process is characterized by the property that its projections to Π_I, $I \subset\subset \mathbb{N}$, are Markov with the semigroup $R_{t,I}$.

A nice direct description of the coalescent process is due to Goldschmidt and Martin [28], which we shortly describe.

We start with an infinite random tree, i.e. a graph with no loops, which has vertex set $\mathbb{N}$. The construction is as follows. We start with the vertex 1 and 2 with a bond between them. Then vertex 3 is with probability 1/2 either attached to 1 or 2. "Attached" means that we draw a bond either to 1 or 2. Then vertex 4 is with probability 1/3 attached either to $1, 2$ or 3, and one proceeds in this way. It is clear that in this way an infinite tree is constructed. We consider 1 to be the root of the tree. The unique path from 1 to vertex $n \neq 1$ is evidently an increasing random sequence of vertices $1 < i_1 < \cdots < i_k = n$. If two bonds i, j are connected by a path $i < i_1 < \cdots i_k = j$, we say that j is below i.

For the dynamics, we have to switch to a formally more general notions, where the vertices are no longer single numbers but subsets of $\mathbb{N}$. The set of all subsets belonging to a tree is supposed to be a partition of $\mathbb{N}$. This notion appears with the dynamics we will describe in a few lines. It will turn out that with probability one, one always falls on infinite partitions where each of the sets is infinite, except at the time 0. We can order the sets in the partition according to the smallest element in the sets, i.e. if A, A' are members of a partition, then $A < A'$ if $\min A < \min A'$. Therefore, from the point of view of the graph, the interpretation of the vertices as sets is just a formal labeling. It is however important for the dynamics and the relation with the coalescent, as we will explain.

We equip now all bonds of the tree with an exponential clock. If a clock rings at time t, then the part of the tree below the bond is erased and the erased sets are added to the vertex above the bond. Of course, as there are infinitely many bonds, there are in any finite time interval infinitely many clocks ringing, and so, it is a priory not clear that in this way a dynamics is defined. This point can however be easily handled by an

approximation with finite trees and a compatibility similar to (4.11). It is then proved in [28] that the dynamics of the partitions, i.e. forgetting the tree structure behind, is identical in law with the dynamics of coalescent process.

The tree representation is particularly useful to prove several crucial properties, like the following one:

Exercise 4.13
Prove that for any $t > 0$, Γ_t has infinitely many countably infinite classes, and no finite classes, almost surely. In particular, Γ_t is non-trivial for any $t > 0$.

Our next task is to relate the above semigroup to the clusterings coming from the Ruelle cascades.

To do that, we describe the semigroup in a different way.

Assume that γ is a partitioning of $\mathbb{N}$, $\gamma = \{C_1, C_2, \ldots\}$, and let $t > 0$. We first choose a PD $\left(\mathrm{e}^{-t}\right)$, leading to a random probability distribution $\bar{\eta} = \{\bar{\eta}_i\}_{i\in\mathbb{N}}$. Conditioned on this realization of the Poisson-Dirichlet process, we choose for every C_k independent random numbers Y_k where

$$P\left(Y_k = j \middle| \bar{\eta}\right) = \bar{\eta}_j.$$

Then we cluster the sets with the same number. This constructs a random partitioning $\gamma' \prec \gamma$. The corresponding kernel is denoted by S_t, i.e. $S_t(\gamma, \cdot)$ is the distribution of the above constructed random γ'.

Lemma 4.14

$$S_t = R_t, \ \forall t \geq 0.$$

Proof. For any finite $I \subset\subset \mathbb{N}$, we can define kernels $S_{t,I}$ in an evident way by restricting the above random matching mechanism to finitely many classes. By the very construction, one has (4.12) satisfied for the kernels S_t, $S_{t,I}$. It therefore suffices to prove $S_{t,I} = R_{t,I}$ for all finite I.

Let $\gamma \in \Pi_I$ have N classes, and γ' be a coarsening obtained by clumping $r_1, \ldots, r_k$ classes together, $\sum r_i = N$. Then, conditioned on $\bar{\eta}$, the probability that under $S_{t,I}$ one arrives at γ'

$$\sum\nolimits^{*}_{i_1,\ldots,i_k} \bar{\eta}_{i_1}^{r_1}\bar{\eta}_{i_2}^{r_2}\cdots\bar{\eta}_{i_k}^{r_k},$$

and so

$$S_{t,I}\left(\gamma,\gamma'\right) = E\sum\nolimits^{*}_{i_1,\ldots,i_k} \bar{\eta}_{i_1}^{r_1}\bar{\eta}_{i_2}^{r_2}\cdots\bar{\eta}_{i_k}^{r_k},$$

the expectation with respect to the Poisson-Dirichlet process PD $\left(\mathrm{e}^{-t}\right)$. This quantity, the reader has (hopefully) computed in Exercise 4.4:

$$E\sum\nolimits^{*}_{i_1,\ldots,i_k} \bar{\eta}_{i_1}^{r_1}\bar{\eta}_{i_2}^{r_2}\cdots\bar{\eta}_{i_k}^{r_k} = \frac{(k-1)!}{(N-1)!}\mathrm{e}^{-t(k-1)}\prod_{i=1}^{k} g\left(r_i, \mathrm{e}^{-t}\right),$$

where $g(r,\zeta)$ is from (4.5). This is exactly the expression, we obtained Proposition 4.11 for $S_{t,I}(\gamma,\gamma')$. ■

We are now in the position to identify the law of the Ruelle-clustering in terms of the coalescent process:

We take as before $0 < \zeta_1 < \cdots < \zeta_K < 1$, and define the clustering $\mathcal{Z}_j,\ 0 \leq j \leq K$, by (4.9).

Theorem 4.15
The law of $(\mathcal{Z}_K, \mathcal{Z}_{K-1}, \ldots, \mathcal{Z}_1)$ *is the same as that of* $\left(\Gamma_0, \Gamma_{t_1}, \ldots, \Gamma_{t_{K-1}}\right)$ *with*

$$\mathrm{e}^{-t_i} = \frac{\zeta_{K-i}}{\zeta_K}.$$

Proof. We first check the case $K = 2$ where there is only one non-trivial partitioning, namely $\mathcal{Z}_1$, and where we have already done the computation in the proof of Proposition 4.7. There we have proved that $\mathcal{Z}_1$ is obtained by attaching marks to $\mathbb{N}$ coming from a PD $\left(\frac{\zeta_1}{\zeta_2}\right)$, and identifying points with the same marks. (See Remark 4.8). So this is exactly the procedure we have for the kernel S_{t_1} with $\mathrm{e}^{-t_1} = \frac{\zeta_1}{\zeta_2}$.

For the general $K \geq 2$ case, the same argument shows that $\mathcal{Z}_{K-1}$ is obtained from the trivial (i.e. non-)clustering $\mathcal{Z}_K$ by applying the kernel S_{t_1} with $\mathrm{e}^{-t_1} = \zeta_{K-1}/\zeta_K$. Now, the way $\mathcal{Z}_{K-2}$ is obtained from $\mathcal{Z}_{K-1}$ is again simply by setting marks to the points of the $(K-1)$-th level, coming from the $(K-2)$-th level, and matching points (i.e. clusters of the finite point process) which have the same marks. This transition is done via the kernel S_{t_2} where $t_2 = \zeta_{K-2}/\zeta_{K-1}$. There is however one difficulty: One has to check that the new clustering is not influenced (stochastically) by the first clustering.

For that, remember that $\mathcal{Z}_{K-1}$ is obtained through a two-stage procedure: One chooses the marks conditionally independent, according to a probability distribution which is computed from $\eta^1, \ldots, \eta^{K-1}$ trough

$$p^{(K-1)}\left(i_1, \ldots, i_{K-1}\right) = \frac{\left(\eta^1_{i_1}\eta^2_{i_1,i_2}\cdots\eta^{K-1}_{i_1,\ldots,i_{K-1}}\right)^{\zeta_K}}{\text{normalization}}.$$

This is a PD (ζ_{K-1}/ζ_K) which is independent of of $\eta^1, \ldots, \eta^{K-2}$. Now, the clustering from $\mathcal{Z}_{K-1}$ to $\mathcal{Z}_{K-2}$ is obtained again through conditionally independent marks, the distribution of the marks being given by

$$p^{(K-2)}\left(i_1, \ldots, i_{K-2}\right) = \frac{\left(\eta^1_{i_1}\eta^2_{i_1,i_2}\cdots\eta^{K-2}_{i_1,\ldots,i_{K-2}}\right)^{\zeta_{K-2}}}{\text{normalization}}$$

which is a PD $(\zeta_{K-2}/\zeta_{K-1})$, depends on η^{K-2}, of course, but is independent of $\eta^1, \ldots, \eta^{K-3}$. Therefore, the clustering procedure from $\mathcal{Z}_{K-1}$ to $\mathcal{Z}_{K-2}$ is independent of the clustering procedure from $\mathcal{Z}_K$ to $\mathcal{Z}_{K-1}$: One just takes the clusters, puts the marks according to $p^{(K-2)}$ which itself is a PD $(\zeta_{K-2}/\zeta_{K-1})$, so that the transition is simply by S_{t_2}. Then again, this clustering is independent of $\eta^1, \ldots, \eta^{K-3}$, and one proceeds in this way. ■

5 Guerra's replica symmetry breaking bound: The Aizenman-Sims-Starr proof

5.1 The Aizenman-Sims-Starr random overlap structures

Definition 5.1
*A **random overlap structure** $\mathcal{R}$ (ROSt for short) consists of a finite or countable set A, a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, and random variables $\eta_\alpha \geq 0$, $q_{\alpha,\alpha'}$, $\alpha, \alpha' \in A$, defined on this probability space, satisfying the following properties*

1. $\sum_\alpha \eta_a < \infty$
2. $\left(q_{\alpha,\alpha'}\right)$ *is positive definite and satisfies* $q_{\alpha,\alpha} = 1$.

The η_α play the rôle of (unnormalized) Gibbs weights, and the q's are the abstract overlaps.

Example 5.2
As an example take $A = \Sigma_N \stackrel{\text{def}}{=} \{-1, 1\}^N$. *For* η_σ, $\sigma \in \Sigma_N$, *we take*

$$\eta_\sigma \stackrel{\text{def}}{=} \exp\left[\frac{\beta}{\sqrt{N}} \sum_{1 \leq i < j \leq N} g_{ij}\sigma_i\sigma_j + h\sum_{i=1}^N \sigma_i\right].$$

For $q_{\sigma,\sigma'}$ *we take the standard overlap* $R_N(\sigma, \sigma')$, *as introduced before. We write* $\mathcal{R}_N^{\text{SK}}$ *for this overlap structure. The* q *here are nonrandom. On the other hand, we can use a (random) reordering of the set* A *by ordering the* η_σ *downwards:* $\eta_1 > \eta_2 > \ldots > \eta_{2^N}$. *After this random reordering, the* q *become random:* $q_{1,2}$ *for instance is the overlap of the two indices with the largest* η*-weight.*

Example 5.3
Another overlap structure is defined by Ruelle's probability cascades introduced in the last section. Fix $0 = \zeta_0 < \zeta_1 < \ldots < \zeta_K < 1$. *We take* $A = \mathbb{N}^K$, *and the* η *are the (unnormalized) weights* $\eta_{\mathbf{i}}$ *as in the last section with* ζ_i, $1 \leq i \leq K$, *see (4.7). The overlaps are defined in the following way. Fix a sequence* $0 \leq q_1 < q_2 < \ldots < q_K < q_{K+1} = 1$, *and we set* $q_{\mathbf{i},\mathbf{i}'} = q_r$ *with*

$$r = r\left(\mathbf{i}, \mathbf{i}'\right) \stackrel{\text{def}}{=} \max\left\{k : (i_1, \ldots, i_k) = \left(i'_1, \ldots, i'_k\right)\right\} + 1,$$

i.e. we measure the hierarchical distance on the tree, and weight it with the function q. *For this random overlap structure, we write* $\mathcal{R}_K^{\text{Ruelle}}$. *Remark that if* $\mathbf{i}, \mathbf{i}'$ *don't overlap, we still give them an overlap* q_1 *which may be positive.*

One may also just take the clustering process of Section 4.3 to define the overlaps, avoiding in this way the necessity to work with finitely many levels.

Given any ROSt, we attach to it families of Gaussian random variables $(y_{\alpha,i})_{\alpha\in A,\ i\in\mathbb{N}}$, $(\kappa_\alpha)_{\alpha\in A}$ by requiring

$$\mathbb{E}\left(\kappa_\alpha\kappa_{\alpha'}\right) = q^2_{\alpha,\alpha'}, \tag{5.1}$$

and the "cavity field" by

$$\mathbb{E}\left(y_{\alpha,j}y_{\alpha',j'}\right) = q_{\alpha,\alpha'}\delta_{j,j'}. \tag{5.2}$$

The κ and the y are independent. In case, the q's itself are random variables, these are just the conditional distributions, given (η, q). It is not difficult to see that such random variables exist. By an extension of the probability space, we can assume that all the random variables are defined on a single probability space. $\mathbb{E}$-expectations below refer to taking the expectations over the overlap structure and then also over these (conditional) Gaussians.

The above notion of a ROSt needs some explanation. The basic idea comes from what in the physics literature is called the "cavity method". We consider the standard SK-Hamiltonian, but now with $N+M$ spins, where one should think of N being much larger than M. We then try to write the Hamiltonian in terms of the Hamiltonian on N spin variables acting on the M "newcomers". We write $\tau_i = \sigma_{N+i}$ for the newcomers.

$$\begin{aligned}
&\frac{\beta}{\sqrt{N+M}}\sum_{1\le i<j\le N+M} g_{ij}\sigma_i\sigma_j + h\sum_{i=1}^{N+M}\sigma_i \\
&= \frac{\sqrt{N}}{\sqrt{N+M}}\frac{\beta}{\sqrt{N}}\sum_{1\le i<j\le N} g_{ij}\sigma_i\sigma_j + h\sum_{i=1}^{N}\sigma_i + \frac{\beta}{\sqrt{N+M}}\sum_{j=1}^{M}\left(\sum_{i=1}^{N} g_{i,N+j}\sigma_i\right)\tau_j \\
&+ \frac{\beta}{\sqrt{N+M}}\sum_{1\le i<j\le M} g_{N+i,N+j}\tau_i\tau_j + h\sum_{j=1}^{M}\tau_j.
\end{aligned}$$

We neglect the parts which are stochastically $o(1)$ for $N\to\infty$, M fixed. In particular, we can neglect the interaction among the newcomers, i.e. we can drop the fourth summand on the right hand side above. Furthermore, we may as well replace $\sqrt{N+M}$ by $\sqrt{N}$ in the third summand. We however should not replace $\frac{\sqrt{N}}{\sqrt{N+M}}$ by 1 in the first summand, because $N^{-1/2}\sum_{i<j\le N} g_{ij}\sigma_i\sigma_j$ is typically of order N under the Gibbs distribution. Defining the cavity variables

$$y_{\sigma,j} \stackrel{\text{def}}{=} \frac{1}{\sqrt{N}}\sum_{j=1}^{M} g_{i,N+j}\sigma_i,$$

we see that they have exactly the right distribution as required in (5.2), with respect to the random overlap structure $\mathcal{R}_N^{\mathrm{SK}}$ coming from the N system. The κ's are used to

correct the normalization in the first summand. Put $U(\sigma) \stackrel{\text{def}}{=} \sum_{1\le i<j\le N} g_{ij}\sigma_i\sigma_j$. Then

$$\begin{aligned}\mathbb{E}\left(\frac{U(\sigma)}{\sqrt{N+M}}\frac{U(\sigma')}{\sqrt{N+M}}\right) &= \frac{N}{(N+M)}\mathbb{E}\left(\frac{U(\sigma)}{\sqrt{N}}\frac{U(\sigma')}{\sqrt{N}}\right)\\ &= \frac{N}{(N+M)}\left(\frac{N}{2}R_N(\sigma,\sigma')^2-\frac{1}{2}\right)\\ &\approx \mathbb{E}\left(\frac{U(\sigma)}{\sqrt{N}}\frac{U(\sigma')}{\sqrt{N}}\right)-\frac{M}{2}q_{\sigma,\sigma'}^2\end{aligned}$$

Therefore,

$$\left\{\frac{U(\sigma)}{\sqrt{N}}\right\}_{\sigma\in\Sigma_N} \approx^{\mathcal{L}} \left\{\frac{U(\sigma)}{\sqrt{N+M}}+\sqrt{\frac{M}{2}}\kappa_\sigma\right\}_{\sigma\in\Sigma_N},$$

with U and κ being independent, up to an error which disappears in the $N\to\infty$ limit, M fixed.

If we set

$$\eta_\sigma \stackrel{\text{def}}{=} \exp\left[\frac{\beta}{\sqrt{N+M}}\sum_{i<j\le N} g_{ij}\sigma_i\sigma_j + h\sum_{i=1}^N \sigma_i\right],$$

we see that

$$\begin{aligned}Z_{N+M} &= \sum\nolimits_{\sigma\in\Sigma_N,\ \tau\in\Sigma_M} \eta_\sigma \exp\left[\sum\nolimits_{i=1}^M (\beta y_{\sigma,i}+h)\tau_i\right],\\ Z_N &\approx \sum\nolimits_{\sigma\in\Sigma_N} \eta_\sigma \exp\left[\beta\sqrt{M/2}\kappa_\sigma\right],\end{aligned}$$

and therefore

$$\frac{Z_{N+M}}{Z_N} \approx \frac{\sum_{\sigma\in\Sigma_N,\ \tau\in\Sigma_M}\eta_\sigma\exp\left[\sum_{i=1}^M(\beta y_{\sigma,i}+h)\tau_i\right]}{\sum_{\sigma\in\Sigma_N}\eta_\sigma\exp\left[\beta\sqrt{M/2}\kappa_\sigma\right]}. \tag{5.3}$$

Here we have used the ROSt from the N-spin SK model (with Gibbs weights coming from a slightly changed temperature parameter). As $\mathbb{E}\log Z_N$ is $\approx Nf(\beta,h)$, it is natural to look at

$$\frac{1}{M}\mathbb{E}\log Z_{N+M} - \frac{1}{M}\mathbb{E}\log Z_M,$$

which then should, at least for large N be close to $f(\beta,h)$. Aizenman, Sims and Starr had the idea to consider the above object when the N system is replaced by an *arbitrary* ROSt $\mathcal{R}$, and therefore to consider the "relative finite M free energy"

$$\begin{aligned}G_M(\beta,h,\mathcal{R}) := \frac{1}{M}\mathbb{E}\log\sum\nolimits_{\alpha,\tau\in\Sigma_M}\eta_\alpha\exp\left[\sum\nolimits_{i=1}^M(\beta y_{\alpha,i}+h)\tau_i\right]\\ -\frac{1}{M}\mathbb{E}\log\sum\nolimits_{\alpha}\eta_\alpha\exp\left[\beta\sqrt{M/2}\kappa_\alpha\right]\end{aligned} \tag{5.4}$$

where the $\mathbb{E}$ expectation is taken with respect both to the law of the random overlap structure and the cavity variables $y_{\alpha,i}$ and κ_α. When taking $\mathcal{R} = \mathcal{R}_N^{\mathrm{SK}}$, one has, up to a negligible corrections when $N \to \infty$:

$$G_M\left(\beta, h, \mathcal{R}_N^{\mathrm{SK}}\right) = \frac{1}{M}\mathbb{E}\log Z_{N+M} - \frac{1}{M}\mathbb{E}\log Z_N. \tag{5.5}$$

Theorem 5.4 (Guerra, Aizenman-Sims-Starr)
For any M, and any random overlap structure $\mathcal{R}$ one has

$$f_M(\beta, h) := \frac{1}{M}\mathbb{E}\log Z_M \leq G_M(\beta, h, \mathcal{R}), \tag{5.6}$$

Z_M here being the SK-partition function.

Proof. To a large extent it is a rerun of the computation done in Section 3.3. One uses the following interpolation:

$$H_M(\tau, \alpha, t) := \frac{\sqrt{1-t}}{\sqrt{M}}\sum\nolimits_{1\leq i<j\leq M} g_{ij}\tau_i\tau_j + \sqrt{\frac{M(1-t)}{2}}\kappa_\alpha + \sqrt{t}\sum\nolimits_{i=1}^{M} y_{\alpha,i}\tau_i$$

and defines the unnormalized Gibbs weights on $A \times \Sigma_M$

$$u(\alpha, \tau, t) := \eta_\alpha \exp\left[\beta H_M(\tau, \alpha, t) + h\sum\nolimits_i \tau_i\right].$$

After normalization, they lead to the Gibbs measure $\mathcal{G}_t$. Then we write $\nu^{(k)}$ for the expectation under $\mathbb{P}\otimes\mathcal{G}_t^{(k)}$, where $\mathbb{P}$ is the probability law, governing the cavity variables $y_{\alpha,i}$, the κ's, the g's, and the q's, if they are random. Remark however, that the g's and the rest are independent, and conditionally on the q's, the y's and the κ's are independent. Put

$$\hat{G}_M(\beta, h, t, \mathcal{R}) := \frac{1}{M}\mathbb{E}\log \sum_{\alpha\in A,\ \tau\in\Sigma_M} u(\alpha, \tau, t) \tag{5.7}$$
$$- \frac{1}{M}\mathbb{E}\log\sum\nolimits_{\alpha\in A}\eta_\alpha \exp\left[\beta\sqrt{M/2}\kappa_\alpha\right].$$

where $\mathbb{E}$ is taken with respect to the overlap structure, i.e. the y's and the κ's, *and* the g's (which are supposed to be independent). For $t = 0$, the κ-part cancels, and one just gets $f_M(\beta, h)$. For $t = 1$, one gets $G_M(\beta, h, \mathcal{R})$.

We compute the t-derivative of $\hat{G}_M$. Remark that the denominator on the right hand side does not depend on t, so it does not appear. We therefore get

$$\frac{d\hat{G}}{dt} = \frac{\beta}{M}\nu_t^{(1)}\left(\frac{dH_M}{dt}\right)$$

$$\frac{dH_M(\tau, \alpha, t)}{dt} = -\frac{1}{2\sqrt{M}\sqrt{1-t}}\sum\nolimits_{1\leq i<j\leq M} g_{ij}\tau_i\tau_j - \sqrt{\frac{M}{2}}\frac{1}{2\sqrt{1-t}}\kappa_\alpha$$
$$+ \frac{1}{2\sqrt{t}}\sum_{i=1}^{M} y_{\alpha,i}\tau_i,$$

so we get

$$\frac{d\hat{G}}{dt} = -S_1 - S_2 + S_3, \text{ say.}$$

The computation of S_1 is exactly the same as in Section 3.3. Remark that the g_{ij} are taken independently of $\mathcal{R}$.

$$S_1 = \frac{\beta^2}{4}\left[1 - \nu_t^{(2)}\left(R_M\left(\tau,\tau'\right)^2\right)\right].$$

Here we stress a bit the notation. $\nu_t^{(2)}$ is a probability measure on $\left(A \times \Sigma_M\right)^2$. When we write $\nu_t^{(2)}\left(R_M\left(\tau,\tau'\right)^2\right)$, we mean in fact that we sum $R_M\left(\tau,\tau'\right)^2$ over $\left(\left(\alpha,\tau\right),\left(\alpha',\tau'\right)\right)$, weighted with the probabilities from $\nu_t^{(2)}$.

We do the same type of computation for S_2. The κ's are *conditionally* Gaussian, conditioned on the q's. This however doesn't matter: We first compute the conditional expectation according to Proposition 1.3, and afterwards integrate out over the distribution of q. The conditional covariances $\operatorname{cov}\left(\kappa_\alpha,\kappa_{\alpha'}\right)$ are $q\left(\alpha,\alpha'\right)^2$, and using the same computation as before, we get

$$S_2 = \frac{\beta^2}{4}\left[1 - \nu_t^{(2)}\left(q\left(\alpha,\alpha'\right)^2\right)\right].$$

Finally, for the S_3 we have Gaussians $\sum_{i=1}^M y_{\alpha,i}\tau_i$ with covariances $q_{\alpha,\alpha'}\sum_{i=1}^M \tau_i\tau_i'$ leading to

$$S_3 = \frac{\beta^2}{2}\left[1 - \nu_t^{(2)}\left(R_M\left(\tau,\tau'\right)q\left(\alpha,\alpha'\right)\right)\right].$$

Therefore

$$-S_1 - S_2 + S_3 = \frac{\beta^2}{4}\nu_t^{(2)}\left(\left[R_M\left(\tau,\tau'\right) - q\left(\alpha,\alpha'\right)\right]^2\right) \geq 0.$$

So, we get

$$\frac{d\hat{G}_M\left(\beta,h,t,\mathcal{R}\right)}{dt} \geq 0, \tag{5.8}$$

and therefore,

$$G_M\left(\beta,h,\mathcal{R}\right) = \hat{G}_M\left(\beta,h,1,\mathcal{R}\right) \geq \hat{G}_M\left(\beta,h,0,\mathcal{R}\right) = f_M\left(\beta,h\right)$$

which immediately implies the theorem. ■

The theorem gives upper bounds for $f_M\left(\beta,h\right)$ by choosing any random overlap structure. Of course, the "correct" choice would be to pick the ROSt from SK, but then, one cannot do any computation. The really interesting bound comes from taking the Ruelle ROSt which gives the Parisi expression as an upper bound, as we see in the next subsection.

One has however to be aware that one is using here a very special property of the SK-model. Even in harmless looking modification (even simplification), it no longer

works. For instance, take an SK-type model where the interaction is only between the first half of the spins and the second half, i.e.

$$H_N(\sigma) = \frac{1}{\sqrt{N}} \sum_{i=1}^{N/2} \sum_{j=N/2+1}^{N} g_{ij}\sigma_i\sigma_j.$$

In that case, the Guerra type bounds do not apply. For the high temperature regime, methods developed by Talagrand do the job, but the low temperature regime is still completely open.

5.2 Guerra's replica symmetry breaking bound

We first have to explain the Parisi formula for the SK-model:

If $f : \mathbb{R} \to \mathbb{R}$ is a function satisfying

$$|f(x)| \leq C\mathrm{e}^{C|x|} \tag{5.9}$$

for some $C > 0$, $0 < \zeta \leq 1$, $\Delta \geq 0$, we define

$$\Psi_{\zeta,\Delta}(f)(x) := \left[Ef(x + \Delta Z)^{\zeta}\right]^{1/\zeta}, \tag{5.10}$$

with a standard Gaussian Z. It is readily checked that the operation $\Psi_{m,\Delta}$ maps the set of functions which satisfy (5.9) into itself.

If $\zeta = (\zeta_0, \zeta_1, \ldots, \zeta_K)$ and $\mathbf{q} = (q_1, \ldots, q_{K+1})$ with

$$0 = \zeta_0 < \zeta_1 < \cdots < \zeta_{K-1} < \zeta_K = 1,$$

$$0 \leq q_1 < q_2 < \cdots < q_K < q_{K+1} = 1,$$

define the Parisi measure with respect to these sequences by

$$\pi_{\zeta,\mathbf{q}} := \sum_{i=1}^{K} (\zeta_i - \zeta_{i-1})\,\delta_{q_i}, \tag{5.11}$$

and put

$$\varphi_{K,\zeta,\mathbf{q}} := \left(\Psi_{\zeta_1,\beta\sqrt{q_2-q_1}} \circ \cdots \circ \Psi_{\zeta_K,\beta\sqrt{q_{K+1}-q_K}}\right)(\cosh).$$

Set

$$\mathcal{P}_K(\zeta, \mathbf{q}; \beta, h) \stackrel{\mathrm{def}}{=} E\log\varphi_{K,\zeta,\mathbf{q}}(h + \beta\sqrt{q_1}Z) - \frac{\beta^2}{4}\sum_{i=1}^{K}\zeta_i\left(q_{i+1}^2 - q_i^2\right) + \log 2. \tag{5.12}$$

Theorem 5.5 (Parisi, Guerra, Talagrand, Panchenko)
For the SK-model free energy $f(\beta, h)$ one has

$$f(\beta, h) = \inf_{K,\zeta,\mathbf{q}} \mathcal{P}_K(\zeta, \mathbf{q}; \beta, h)$$

Remark 5.6
Let's look at $\mathcal{P}_1$. There is only one parameter $q = q_1$ which is assumed to satisfy $0 \leq q < 1$. Then φ_1 is

$$\begin{aligned}\varphi_1(x) &= \Psi_{1,1-q}(\cosh)(x) = E_Z \cosh\left(x + \beta\sqrt{1-q}Z\right) \\ &= \cosh(x) \exp\left[\frac{\beta^2}{2}(1-q)\right].\end{aligned}$$

Therefore,

$$\begin{aligned}\mathcal{P}_1(\zeta, \mathbf{q}; \beta, h) &= E_Z \log\cosh(h + \beta\sqrt{q}Z) + \frac{\beta^2}{2}(1-q) - \frac{\beta^2}{4}\left(1-q^2\right) + \log 2 \\ &= E_Z \log\cosh(h + \beta\sqrt{q}Z) + \frac{\beta^2}{4}(1-q)^2 + \log 2,\end{aligned}$$

and the infimum is over q yields $\mathrm{RS}(\beta, h)$.

Guerra first proved the one-sided bound which follows from Theorem 5.4 and

Lemma 5.7
We take $\mathcal{R}_K^{\mathrm{Ruelle}}(\zeta, \mathbf{q})$ as the ROSt from Example 5.3. Then

$$G_M\left(\beta, h, \mathcal{R}_K^{\mathrm{Ruelle}}\right) = G_1\left(\beta, h, \mathcal{R}_K^{\mathrm{Ruelle}}\right) = \mathcal{P}_K(\zeta, \mathbf{q}; \beta, h). \tag{5.13}$$

Remark 5.8
There is a slight problem with the formulation above, as $\mathcal{R}_K^{\mathrm{Ruelle}}$ is defined only for $\zeta_K < 1$. There is however no problem to define $\mathcal{P}_K(\zeta, q; \beta, h)$ with $\zeta_K < 1$, and let $\zeta_K \to 1$ in the end.

Proof. We handle the two parts in (5.4) separately. We take $M = 1$. It will be clear after the computation that for general M the outcome will be the same. We first represent the cavity variables in a convenient way:

$$y_{\mathbf{i}} = \sqrt{q_1} g^{(0)} + \sum_{k=1}^{K} \sqrt{q_{k+1} - q_k}\, g^{(k)}_{i_1,\ldots,i_k}, \tag{5.14}$$

where the new g's are independent standard Gaussian, also independent of the g_{ij}. As we assume $M = 1$, we need only one set of the $y_{\mathbf{i}}$'s. For general M, we would need M independent copies of them. The $\kappa_{\mathbf{i}}$ are constructed in a similar way.

$$\begin{aligned}\frac{1}{2}\sum_{\mathbf{i},\tau\in\Sigma_1} \eta_{\mathbf{i}} \exp\left[(\beta y_{\mathbf{i}} + h)\tau\right] &= \sum_{\mathbf{i}} \eta_{\mathbf{i}} \cosh(\beta y_{\mathbf{i}} + h) \\ &= \sum_{(i_1,\ldots,i_K)} \eta^1_{i_1} \eta^2_{i_1 i_2} \cdot \cdots \cdot \eta^K_{i_1 i_2 \ldots i_K} \cosh(X_{K,\mathbf{i}} + h),\end{aligned}$$

with the abbreviation

$$X_l := \beta\sqrt{q_1}g^{(0)} + \beta\sum_{k=1}^{l}\sqrt{q_{k+1}-q_k}g^{(k)}_{i_1,\ldots,i_k},\ 0 \le l \le K.$$

For $l = 0$, the second summand is absent.

We write

$$\mathcal{F}_l := \sigma\left(\eta^1_{i_1}, \eta^2_{i_1 i_2}, \ldots, \eta^l_{i_1 i_2 \ldots i_l}, g^{(0)}, \ldots, g^{(l)}\right)$$

We condition on $\mathcal{F}_{K-1}$. Then the point process $\left\{\left(\eta^K_{i_1 i_2 \ldots i_K}\cosh\left(X_K + h\right)\right)_{i_K}\right\}$ is a PPP $\left(t \to \zeta_K t^{-\zeta_K - 1}\right)$ whose points are multiplied by the independent random variables $\left(\cosh\left(\beta\sum_{n=0}^{K}\sqrt{q_{k+1}-q_k}g^{(n)}_{i_1,\ldots,i_n} + h\right)\right)_{i_K}$.From Proposition 4.1 a) the conditional law (conditioned on $\mathcal{F}_{K-1}$) of

$$\left\{\eta^K_{i_1 i_2 \ldots i_K}\cosh\left(X_K + h\right)\right\}_{i_K}$$

is the same as that of

$$\left\{\left[\mathbb{E}\left(\cosh^{\zeta_K}\left(X_K + h\right)|\mathcal{F}_{K-1}\right)\right]^{1/\zeta_K}\eta^K_{i_1,\ldots,i_K}\right\}_{i_K},$$

and

$$\mathbb{E}\left(\cosh^{\zeta_K}\left(X_K + h\right)|\mathcal{F}_{K-1}\right) = \Psi_{\zeta_K, q_{K+1}-q_K}\left(\cosh\right)\left(h + X_{K-1}\right),$$

by the definition of Ψ. So, we have

$$\left\{\eta_{\mathbf{i}}\cosh\left(\beta y_{\mathbf{i}} + h\right)\right\}_{\mathbf{i}} =^{\mathcal{L}} \eta^1_{i_1}\cdots\eta^{K-1}_{i_1,\ldots,i_{K-1}}\left[\Psi_{\zeta_K, q_{K+1}-q_K}\left(\cosh\right)\left(h + X_{K-1}\right)\right]\eta^K_{i_1,\ldots,i_K}.$$

Arguing in exactly the same way, we see that

$$\left\{\eta^{K-1}_{i_1,\ldots,i_{K-1}}\Psi_{\zeta_K, q_{K+1}-q_K}\left(\cosh\right)\left(h + X_{K-1}\right)\right\}_{i_{K-1}},$$

is in law identical to

$$\left\{\Psi_{\zeta_{K-1}, q_K - q_{K-1}}\circ\Psi_{\zeta_K, q_{K+1}-q_K}\left(\cosh\right)\left(h + X_{K-2}\right)\eta^{K-1}_{i_1 i_2 \ldots i_{K-1}}\right\}_{i_{K-1}},$$

and therefore,

$$\begin{aligned}\left\{\eta_{\mathbf{i}}\cosh\left(\beta y_{\mathbf{i}} + h\right)\right\}_{\mathbf{i}} =^{\mathcal{L}}\ & \eta^1_{i_1}\eta^2_{i_1 i_2}\cdots\eta^{K-2}_{i_1,\ldots,i_{K-2}} \\ & \left[\Psi_{\zeta_{K-1}, q_K - q_{K-1}}\circ\Psi_{\zeta_K, q_{K+1}-q_K}\left(\cosh\right)\left(h + X_{K-2}\right)\right] \\ & \times\eta^{K-1}_{i_1,\ldots,i_{K-1}}\eta^K_{i_1,\ldots,i_K}.\end{aligned}$$

In this way, one proceeds, and arrives at

$$\left\{\eta_{\mathbf{i}}\cosh\left(\beta y_{\mathbf{i}} + h\right)\right\}_{\mathbf{i}} =^{\mathcal{L}} \left\{\varphi_{K,\zeta,\mathbf{q}}\left(h + \sqrt{q_1}g^{(0)}\right)\eta_{\mathbf{i}}\right\}_{\mathbf{i}}.$$

The second part in (5.4) is similar and somewhat simpler because there one has in every step just an integration of a Gaussian in the exponent. If $\{\eta_k\}$ is a PPP $\left(\zeta t^{-\zeta-1}dt\right)$, and the Z_k are independent Gaussians with variance $q_{i+1}^2 - q_i^2$, then

$$\left\{\eta_i \exp\left[\left(\beta/\sqrt{2}\right) Z_i\right]\right\}_i =^{\mathcal{L}} \left\{\left[E\left(\exp\left[\zeta\left(\beta/\sqrt{2}\right) Z\right]\right)\right]^{1/\zeta} \eta_i\right\}_i,$$

and

$$\begin{aligned}\left[E\left(\exp\left[\zeta\left(\beta/\sqrt{2}\right) Z\right]\right)\right]^{1/m} &= \left[\exp\left[\zeta^2 \frac{\beta^2}{4}\left(q_{i+1}^2 - q_i^2\right)\right]\right]^{1/\zeta} \\ &= \exp\left[\zeta \frac{\beta^2}{4}\left(q_{i+1}^2 - q_i^2\right)\right].\end{aligned}$$

Iterating this in the same way as above, we see that multiplying the points $\eta_{\mathbf{i}}$ by

$$\exp\left[\left(\beta/\sqrt{2}\right)\kappa_{\mathbf{i}}\right]$$

simply leads to a multiplication of the point process by $\exp\left[\left(\beta^2/4\right)\sum_{i=1}^K \zeta_i\left(q_{i+1}^2 - q_i^2\right)\right]$.

We implement that now into (5.4) with $M = 1$

$$\begin{aligned}G_1\left(\beta, h, \mathcal{R}^{\text{Ruelle}}\right) &= \mathbb{E}\left(\log \sum\nolimits_{\mathbf{i},\tau\in\{-1,1\}} \eta_{\mathbf{i}} \exp\left[\left(\beta y_{\mathbf{i}} + h\right)\tau\right]\right) \\ &- \mathbb{E}\left(\log \sum\nolimits_{\mathbf{i}} \eta_{\mathbf{i}} \exp\left[\beta\sqrt{1/2}\kappa_{\mathbf{i}}\right]\right) \\ &= \mathbb{E}\left(\log\left(\varphi_{K,\zeta,\mathbf{q}}\left(h + \sqrt{q_1}g^{(0)}\right)\sum\nolimits_{\mathbf{i}} \eta_{\mathbf{i}}\right)\right) \\ &- \mathbb{E}\left(\log\left(\exp\left[\left(\beta^2/4\right)\sum_{i=0}^K \zeta_i\left(q_{i+1}^2 - q_i^2\right)\right]\sum\nolimits_{\mathbf{i}} \eta_{\mathbf{i}}\right)\right) + \log 2 \\ &= \mathbb{E}\left(\log \varphi_{K,\zeta,\mathbf{q}}\left(h + \sqrt{q_1}g^{(0)}\right)\right) - \frac{\beta^2}{4}\sum_{i=0}^K \zeta_i\left(q_{i+1}^2 - q_i^2\right) + \log 2\end{aligned}$$

The upshot of this computation is that

$$\begin{aligned}G_1\left(\beta, h, \mathcal{R}\right) &= E\log Y_1 - \frac{\beta^2}{4}\sum_{i=0}^K \zeta_i\left(q_{i+1}^2 - q_i^2\right) + \log 2 \\ &= \mathcal{P}_K\left(\zeta, \mathbf{q}; \beta, h\right).\end{aligned}$$

It is fairly evident from this computation that we get the same for arbitrary M. (One is just having M factors of $\cosh(\cdot)$ with independent contents, so in every step of the above argument, the factoring remains). ■

Combining this result with Theorem 5.4, one gets Guerra's result:

Theorem 5.9 (Guerra)

$$f_M(\beta, h) \leq \mathcal{P}_K(\zeta, \mathbf{q}; \beta, h)$$

for any K, and any sequence ζ and $\mathbf{q}$. Therefore

$$f_M(\beta, h) \leq \inf_{K,\zeta,\mathbf{q}} \mathcal{P}_K(\zeta, \mathbf{q}; \beta, h).$$

Remark 5.10
The variational formula on the right hand side is analytically difficult. For small β, one knows that one can take $K = 1$ which leads to the replica symmetric formula we had proved in Section 3.4. In physics literature, it is claimed that this is the situation up to the AT-line, but this has not been proved. It is also claimed that beyond the AT-line, one has to let $K \to \infty$. That isn't proved neither. One can formulate the variational formula directly in the continuum with an arbitrary Parisi measure not just one in the form (5.11). It has recently been proved by Auffinger and Chen [7] that the variational formula always has a unique solution, but it is not shown that one has to go outside the class of measures of the form (5.11). It is however proved by Toninelli [43] that $K = 1$ will not do beyond the AT-line

6 The Ghirlanda-Guerra identities

I discuss here shortly the Ghirlanda-Guerra identities for the SK-model. These enter into the recent proof of ultrametricity by Panchenko, but in a somewhat complicated way. The difficult point is that the SK Gibbs distribution does not satisfy the identities. This follows from Talagrand's positivity result: The Ghirlanda-Guerra identities imply positivity of the overlaps which is not the case for SK, at least not for $h = 0$, due to symmetry. In order that they can be proved one needs a (small) perturbation of the Hamiltonian which however does not affect the free energy. The Ghirlanda-Guerra identities in this framework were first proved in [27], [2], and variants were proved by many other authors.

One of the basic insights Guerra had was that one should use stability properties of the system under perturbations. The background of this idea was that the Parisi solution itself has quite some stability properties. Therefore, the hope was that if one can prove that the systems under investigation have similar stability properties, in the $N \to \infty$ limit, then one can derive properties which could in the end prove the validity of the Parisi solution. This was the key idea for the derivation of the Ghirlanda-Guerra identities by just using basic stability properties. At first, it was generally believed that these identities are not sufficient to prove the Parisi formula, but in the end it turned out that this was wrong. For a discussion of the background, see also the insightful paper by Aizenman and Contucci [2].

We follow here Talagrand's presentation in the second volume of [42], Chapter 12.

We will need the following result about convex functions:

Exercise 6.1
Let f, g be convex function on $\mathbb{R}$, $x \in \mathbb{R}$ and $y > 0$. Then

$$\left|f'(x) - g'(x)\right| \leq g'(x+y) - g'(x-y) + w(x,y)$$

with

$$w(x,y) := y^{-1} \sum_{m=-1}^{1} \left|f(x+my) - g(x+my)\right|.$$

We write H for the SK-Hamiltonian including the parameters N, β, h which we don't explicitly write in the notation. So

$$H(\sigma) := \frac{\beta}{\sqrt{N}} \sum_{1 \leq i < j \leq N} g_{ij}\sigma_i\sigma_j + h\sum_{i=1}^{N} \sigma_i.$$

The Ghirlanda-Guerra identities are proved only under a slightly perturbed Hamiltonian where we keep the perturbation open for the moment:

$$H_x^{\text{per}}(\sigma) = H(\sigma) + xG(\sigma),$$

where $x \in \mathbb{R}$ and $\{G(\sigma)\}_{\sigma \in \Sigma_N}$ is a family of centered Gaussians which is independent of H. We write $\mathcal{G}_x, \mathcal{G}_x^{(n)}, \nu_x$ with the usual meaning, with the perturbed Hamiltonian. The basic property used to derive the identities is the following concentration property:

Proposition 6.2
Assume that $\delta_N^2 := \text{var}(G(\sigma))$ is independent of σ, and

$$\delta_N^2 = O(N), \ \delta_N^2/\sqrt{N} \to \infty. \tag{6.1}$$

Then, for some constant $C(\beta) > 0$, not depending on h, N, and N large enough

$$\int_{-1}^{1} \nu_x\left(\left|\frac{G}{N} - \frac{\nu_x(G)}{N}\right|\right) dx \leq C(\beta)\,\delta_N N^{-3/4}$$

Proof.

$$\nu_x\left(\left|\frac{G}{N} - \frac{\nu_x(G)}{N}\right|\right) \leq \nu_x\left(\left|\frac{G}{N} - \frac{\mathcal{E}_x(G)}{N}\right|\right) + \nu_x\left(\left|\frac{\mathcal{E}_x(G)}{N} - \frac{\nu_x(G)}{N}\right|\right). \tag{6.2}$$

Let $\theta(x) := N^{-1}\log Z_x$ which is a random convex function, and $\psi(x) := \mathbb{E}\theta(x)$. A simple computation gives

$$\theta'(x) = \mathcal{E}_x\left(\frac{G}{N}\right), \ \theta''(x) = \frac{1}{N}\,\text{var}_{\mathcal{G}_x}(G),$$

and therefore

$$\psi''(x) = \frac{1}{N}\nu_x\left((G - \mathcal{E}_x(G))^2\right).$$

So, we obtain

$$\int_{-1}^{1} \nu_x \left(\left(\frac{G}{N} - \frac{\mathcal{E}_x(G)}{N} \right)^2 \right) dx = \frac{1}{N} \left(\psi'(1) - \psi'(-1) \right).$$

By Proposition 1.3, we have

$$\psi'(z) = \nu_z \left(\frac{G}{N} \right) = z \left(\nu_z (U_{1,1}) - \nu_z^{(2)} (U_{1,2}) \right),$$

where $U_{i,k} := N^{-1} \mathbb{E} G\left(\sigma^i\right) G\left(\sigma^k\right)$. Evidently $|U_{i,k}| \leq N^{-1} \sqrt{\mathbb{E} G^2(\sigma^i)\, \mathbb{E} G^2(\sigma^k)} = N^{-1} \delta_N^2$, so

$$\left| \psi'(z) \right| \leq \frac{2|z| \delta_N^2}{N}, \tag{6.3}$$

$$\int_{-1}^{1} \nu_x \left(\left(\frac{G}{N} - \frac{\mathcal{E}_x(G)}{N} \right)^2 \right) dx \leq 4N^{-2} \delta_N^2,$$

$$\int_{-1}^{1} \nu_x \left(\left| \frac{G}{N} - \frac{\mathcal{E}_x(G)}{N} \right| \right) dx \leq \sqrt{2} \sqrt{4N^{-2} \delta_N^2} = \frac{\sqrt{8}}{N} \delta_N = o\left(\delta_N N^{-3/4} \right)$$

For the estimate of the second summand in (6.2), more precisely its integral over x, we represent it as

$$\mathbb{E} \int_{-1}^{1} \left| \theta'(x) - \psi'(x) \right| dx.$$

θ is a random convex function in x, and ψ its convex expectation. We first have to estimate $\mathbb{E} |\theta(x) - \psi(x)|$. $\theta(x)$ is the finite N free energy, and $\psi(x)$ its expectation. We use the concentration inequality in Theorem 1.1 much in the same way as in the proof of Theorem 3.1. This yields that for some constant $C > 0$, we have

$$\mathbb{E} |\theta(x) - \psi(x)| \leq C \frac{\sqrt{\beta^2 N + x^2 \delta_N^2}}{N}.$$

We will use that for x stays bounded, and $\delta_N = O(N)$, and therefore, with some new $C(\beta)$

$$\mathbb{E} |\theta(x) - \psi(x)| \leq \frac{C(\beta)}{\sqrt{N}},$$

We apply now the exercise with $f = \theta$, $g = \psi$, $|x| \leq 1$, and $y = \sqrt{\beta} \delta_N^{-1} N^{1/4}$. As we assume (6.1), $y \leq 1$ for large enough N. Therefore, we get

$$\mathbb{E} w(x, y) \leq C(\beta) / y \sqrt{N}.$$

So, using again (6.3)

$$\begin{aligned}\int_{-a}^{a}\mathbb{E}\left|\theta'(x)-\psi'(x)\right|dx &\leq \frac{C(\beta)}{y\sqrt{N}}+\int_{-1}^{1}\left(\psi'(x+y)-\psi'(x-y)\right)dx\\ &=\frac{C(\beta)}{y\sqrt{N}}+\int_{1-y}^{1+y}\psi'(x)\,dx+\int_{-1-y}^{-1+y}\psi'(x)\,dx.\\ &\leq C(\beta)\left(\frac{1}{y\sqrt{N}}+\frac{|y|\,\delta_N^2}{N}\right),\end{aligned}$$

and by our choice of y, we get that this is bounded by $C(\beta)\,\delta_N N^{-3/4}$. So, we have proved the claim. ■

We apply this concentration inequality to a replicated system with $H_x^{\mathrm{per}}(\sigma)$. As usual, we write ν_x also for $\int \mathcal{G}_x^{(n)} d\mathbb{P}$ with unspecified n. For two replicas σ^i, σ^j, we write

$$U_{ij} := \frac{1}{N}\mathbb{E}G\left(\sigma^i\right)G\left(\sigma^j\right),$$

Let also f be a function $\Sigma_N^n \to \mathbb{R}$ which is bounded by 1.

Theorem 6.3
Let

$$\delta_x := \left|\nu_x\left(U_{1,n+1}f\right)-\frac{1}{n}\sum_{j=2}^{n}\nu_x\left(U_{1,j}f\right)-\frac{1}{n}\nu_x\left(U_{1,2}\right)\nu_x(f)\right|.$$

Then, for some constant $C(\beta,n)>0$

$$\int_{-1}^{1}\delta_x dx \leq C(\beta,n)\,\delta_N N^{-3/4}\log N.$$

Proof. We write

$$\Delta_x := \nu_x\left(\left|\frac{G}{N}-\nu_x\left(\frac{G}{N}\right)\right|\right).$$

Then

$$\eta_x := \nu_x\left(\frac{G}{N}f\right)-\nu_x\left(\frac{G}{N}\right)\nu_x(f)$$

satisfies $|\eta_x|\leq\Delta_x$, as f is assumed to be bounded by 1. Proposition 1.3 gives

$$\begin{aligned}\nu_x\left(\frac{G}{N}\right) &= x\left(\nu\left(U_{11}\right)-\nu\left(U_{12}\right)\right),\\ \nu_x\left(\frac{G}{N}f\right) &= x\left(\sum\nolimits_{j=1}^{n}\nu_x\left(U_{1,j}f\right)-n\nu_x\left(U_{1,n+1}f\right)\right).\end{aligned}$$

By our assumption, U_{11} does not depend on σ. Therefore $\nu_x\left(U_{11}f\right)=\nu_x\left(U_{11}\right)\nu_x(f)$, and we obtain

$$\delta_x \leq \frac{|\eta_x|}{n\,|x|}\leq\frac{\Delta_x}{n\,|x|}\leq\frac{C(\beta,n)\,\delta_N}{|x|\,N^{3/4}}.$$

On the other hand, for any $|x| \leq 1$, we have $\delta_x \leq C\delta_N^2$, and so

$$\delta_x \leq C(\beta, n) \min\left(\delta_N N^{-3/4} |x|^{-1}, \delta_N^2\right).$$

For $|x| \leq \left(\delta_N N^{3/4}\right)^{-1}$, the minimum is $C\delta_N^2$ and

$$\int_{-\left(\delta_N N^{3/4}\right)^{-1}}^{\left(\delta_N N^{3/4}\right)^{-1}} \delta_x dx \leq 2C(\beta, n) \delta_N N^{-3/4}.$$

For $|x| > \left(\delta_N N^{3/4}\right)^{-1}$, we estimate

$$\begin{aligned}\int_{\left(\delta_N N^{3/4}\right)^{-1} \leq |x| \leq 1} \delta_x dx &\leq 2C(\beta, n) \delta_N N^{-3/4} \log\left(\delta_N N^{3/4}\right) \\ &\leq C'(\beta, n) \delta_N N^{-3/4} \log N,\end{aligned}$$

with some new C'. This proves the claim. ■

The interesting feature is that one can apply that to small perturbations which don't influence the free energy of the model. Take for instance

$$G(\sigma) = \varepsilon_N \sum_{k=1}^{N} g_k \sigma_k$$

with $\varepsilon_N \to 0$. Then $\delta_N^2 = \varepsilon_N^2 N$, and

$$U_{ij} = \frac{\varepsilon_N^2}{N} \mathbb{E} \sum_{k=1}^{N} g_k^2 \sigma_k^{(i)} \sigma_k^{(j)} = \varepsilon_N^2 R_{ij}.$$

If f is a bounded function of $R^{(n)} = \left(R_{ij}\right)_{i,j \leq n}$, we get

$$\begin{aligned}\int_{-1}^{1} \Big| \nu_x\left(R_{1,n+1} f\left(R^{(n)}\right)\right) - \frac{1}{n} \sum_{j=2}^{n} \nu_x\left(R_{1,j} f\left(R^{(n)}\right)\right) \\ - \frac{1}{n} \nu_x(R_{1,2}) \nu_x\left(f\left(R^{(n)}\right)\right) \Big| dx \leq C(\beta, n) \varepsilon_N^{-1} N^{-1/4}.\end{aligned}$$

For instance, if $\varepsilon_N = N^{-1/8}$, this still converges to 0. On the other hand:

Exercise 6.4
For the Hamiltonian

$$H_x^{\mathrm{per}}(\sigma) := \frac{\beta}{\sqrt{N}} \sum_{1 \leq i < j \leq N} g_{ij} \sigma_i \sigma_j + h \sum_{i=1}^{N} \sigma_i + x N^{-1/8} \sum_{i=1}^{N} g_i \sigma_i,$$

one has

$$\lim_{N \to \infty} \frac{1}{N} \log Z_{N,x}^{\mathrm{per}} = f(\beta, h),$$

i.e. the free energy is the same as that for SK-model.

With other (small) perturbations, one can prove the Ghirlanda-Guerra identities. If one takes

$$G(\sigma) = \varepsilon_N N^{-p/2+1} \sum_{i_1,i_2,\cdots,i_p \leq N} g_{i_1 i_2 \cdots i_p} \sigma_{i_1} \sigma_{i_2} \cdot \cdots \cdot \sigma_{i_p}$$

with i.i.d. standard Gaussians $g_{i_1 i_2 \cdots i_p}$, one gets $U_{ij} = \varepsilon_N^2 R_{ij}^p$ and therefore, with $\varepsilon_N = N^{-1/8}$

$$\int_{-1}^{1} \left| \nu_x \left(R_{1,n+1}^p f\left(R^{(n)} \right) \right) - \frac{1}{n} \sum_{j=2}^{n} \nu_x \left(R_{1,j}^p f\left(R^{(n)} \right) \right) \right.$$
$$\left. - \frac{1}{n} \nu_x \left(R_{1,2}^p \right) \nu_x \left(f\left(R^{(n)} \right) \right) \right| dx \leq C(p, \beta, n) N^{-1/8}.$$

With the help of these identities one can prove that suitably perturbed Hamiltonians satisfy the Ghirlanda-Guerra identities in the $N \to \infty$ limit, with using perturbations which don't affect the free energy. For more details, see [42] Vol II, Ch. 12.

7 Ultrametricity

A somewhat vaguely formulated claim in the physics literature is that the Gibbs measure is organized in a hierarchical way, and in the end by a Ruelle cascade. A concrete rigorous picture is lacking, but the picture should be as follows. The configuration space Σ_N can be divided into countably many "pure states". This of course does not make any sense, and has always to keep in mind, that in some sense, which is not made precise, this refers to $N \to \infty$. The pure states are then collections A_α of configurations, and the Gibbs measure, restricted to any of the A_α should have the following property: The mean of the σ_i under the Gibbs measure, conditioned on A_α is m_i^α which satisfies the TAP equation. The σ_i, under the conditioned law, are essentially independent. In particular, this means that two replicas σ_i^1, σ_i^2 under the conditioned law, have an overlap $N^{-1} \sum (m_i^\alpha)^2$ which should be a constant $q_{\max} < 1$.[9] The Gibbs weights of the A_α are distributed according to a PD point process, i.e. the distribution is *random.*

A metric d is called an *ultrametric*, if it satisfies the stronger triangle inequality

$$d(x, z) \leq \max\left(d(x, y), d(y, z) \right).$$

An equivalent formulation is that if two balls (with respect to the metric) overlap, then one is contained in the other. An example is the case of a rooted tree with positive weights on the bonds, with the metric space given by the leafs and the weighted graph distance. That's essentially the only example, at least in the case of a metric space with finitely many elements: An ultrametric space is a space with a tree structure.

The general formulation of the ultrametricity conjecture for spin glasses, say with Gaussian Hamiltonians $H_N(\sigma)$, $\sigma \in \Sigma_N$, would be that the metric

$$d\left(\sigma, \sigma'\right) := \left\| H_N(\sigma) - H_N\left(\sigma'\right) \right\|_2$$

[9] This has not to be taken completely literally, as evidently, there is a symmetry between σ and $-\sigma$, at least if $h = 0$, and therefore negative overlaps are as likely as positive ones.

is asymptotically an ultrametric. Take the SK-model without external field where

$$\mathbb{E}\left(H_N(\sigma) - H_N(\sigma')\right)^2 = N\left(1 - R_N(\sigma,\sigma')^2\right).$$

The ultrametricity conjecture would then say that for any $\varepsilon > 0$

$$\lim_{N\to\infty} \nu_N^{(3)}\left\{\left(\sigma^1,\sigma^2,\sigma^3\right) : d\left(\sigma^1,\sigma^3\right) \geq \max\left(d\left(\sigma^1,\sigma^3\right), d\left(\sigma^1,\sigma^3\right)\right) + \varepsilon N\right\} = 0. \quad (7.1)$$

(As the distances are typically of order N, one should ask for deviations from the ultrametricity property by εN). In this form, it is not a proved statement for SK, and I don't know if everybody agrees that it should be correct.

7.1 A non-hierarchical version of the generalized random energy model

A simple case where ultrametricity has been proved is a non-hierarchical version of the generalized random energy model which was investigated in two papers with Nicolas Kistler: [13], [14].

The motivation was coming from the following trivial observation: In the SK-model with N even, $N = 2M$, and $h = 0$, one can split the Hamiltonian:

$$H_N(\sigma) = \frac{1}{\sqrt{N}} \sum_{1\leq i<j\leq M} g_{ij}\sigma_i\sigma_j + \frac{1}{\sqrt{N}} \sum_{M+1\leq i<j\leq N} g_{ij}\sigma_i\sigma_j + \frac{1}{\sqrt{N}} \sum_{1\leq i\leq M} \sum_{M+1<j\leq N} g_{ij}\sigma_i\sigma_j,$$

and the three parts are independent. In an analogy, a random energy version of such a structure would look like

$$H((\alpha_1,\alpha_2)) = X^1_{\alpha_1} + X^2_{\alpha_2} + X^{1,2}_{\alpha_1,\alpha_2}$$

where $1 \leq \alpha_1, \alpha_2 \leq 2^{N/2}$, and the $X^1, X^2, X^{1,2}$ are independent Gaussians with variances of order N, perhaps not the same for the three different types. Say $\mathrm{var}\left(X^1\right) = \kappa_1^2 N$, $\mathrm{var}\left(X^2\right) = \kappa_2^2 N$, and $\mathrm{var}\left(X^{1,2}\right) = \kappa_{1,2}^2 N$. The covariances are then

$$\frac{1}{N}\mathbb{E}\left[H(\alpha_1,\alpha_2)\, H\left(\alpha_1',\alpha_2'\right)\right] = \begin{cases} \kappa_1^2 + \kappa_2^2 + \kappa_{12}^2 & \text{if } \alpha_1 = \alpha_1',\ \alpha_2 = \alpha_2' \\ \kappa_1^2 & \text{if } \alpha_1 = \alpha_1',\ \alpha_2 \neq \alpha_2' \\ \kappa_2^2 & \text{if } \alpha_1 \neq \alpha_1',\ \alpha_2 = \alpha_2' \\ 0 & \text{otherwise} \end{cases}.$$

For this model, the L_2 distance

$$d\left((\alpha_1,\alpha_2),\left(\alpha_1',\alpha_2'\right)\right) := \sqrt{\mathbb{E}\left(H(\alpha_1,\alpha_2) - H\left(\alpha_1',\alpha_2'\right)\right)^2}$$

is evidently not an ultrametric.

It however turns out that the limiting Gibbs distribution is for all choices of $\kappa_1, \kappa_2, \kappa_{12} > 0$ a Ruelle cascade, and the metric d is asymptotically an ultrametric in the sense of (7.1). This is essentially based on simple combinatorial facts which reveal that non-hierarchically organized configurations are less likely than hierarchically organized ones. That is for such type of models not difficult to see, but a similar combinatorial insight is lacking for more complicated spin glasses.

I will shortly describe the approach for these non hierarchical GREMs.

Let's fix the general setup. We fix $K \in \mathbb{N}$, write $[K] := \{1, \dots, K\}$ and consider $\alpha = (\alpha_1, \dots, \alpha_K)$, where $1 \leq \alpha_i \leq 2^{N/K}$. For notational simplicity, we assume that N/K is an integer. K will be fixed through all considerations and $N \to \infty$. For $J \subset [K]$ and α is as above, we write α_J for $(\alpha_i)_{i \in J}$. We write $\Sigma_{[K]}$ for the set of all configurations, and Σ_J for the set of configurations with indices in J. Evidently $|\Sigma_J| = 2^{N|J|/K}$.

Let $\kappa_J^2 \geq 0$, $J \subset [K]$, $J \neq \emptyset$, be numbers satisfying $\sum_{J \in \mathcal{S}} \kappa_J^2 = 1$. The latter is just a normalization of no importance. We set

$$\mathcal{P} := \left\{ J \subset [K] : \kappa_J^2 > 0 \right\}.$$

The model is described by the following Hamiltonian

$$H(\alpha) := \sum_{J \in \mathcal{P}} X_{\alpha_J}^J,$$

where for any $J \in \mathcal{P}$, the $X_{\alpha_J}^J$ are i.i.d. centered Gaussians with variance κ_J^2. For different J's, the variables are independent.

We define as usual

$$f(\beta) := \lim_{N \to \infty} \frac{1}{N} \log Z_{N,\beta} = \frac{1}{N} \mathbb{E} \log Z_{N,\beta},$$

where

$$Z_{N,\beta} := \sum_{\alpha} \exp\left[\beta H(\alpha)\right]$$

The example we had before has $K = 2$ and $\mathcal{P}$ all non-void subsets of [2]. Another exemple is $K = 3$ and $\mathcal{P} = \{\{1,2\}, \{2,3\}, \{1,3\}\}$ in which case

$$H(\alpha_1, \alpha_2, \alpha_3) = X_{\alpha_1, \alpha_2}^{1,2} + X_{\alpha_1, \alpha_3}^{1,3} + X_{\alpha_2, \alpha_3}^{2,3} \tag{7.2}$$

with independent Gaussian X contributions with the assigned variances.

It is clear that Derrida's REM and GREM are special cases. The REM takes $K = 1$, and the GREM has $K > 1$ and

$$\mathcal{P} = \{\{1\}, \{1,2\}, \dots, \{1, \dots, K\}\}.$$

Actually, any **nested** $\mathcal{P}$ leads to a GREM. We call $\mathcal{P}$ nested if $\mathcal{P} = \{J_1, J_2, \dots, J_m\}$ with

$$J_1 \subset J_2 \subset \cdots \subset J_m.$$

Any of our models can be coarse grained into GREMs in the following way: Choose a sequence $\mathbf{T} = \{A_0, \ldots, A_m\}$, $A_i \subset [K]$, with

$$A_0 = \emptyset \subset A_1 \subset A_2 \subset \cdots \subset A_m = [K]\,. \tag{7.3}$$

The A_i are not required to belong to $\mathcal{P}$. Write

$$\hat{\kappa}^2_{A_j} := \sum_{J \in \mathcal{P}_{A_j} \setminus \mathcal{P}_{A_{j-1}}} \kappa^2_J,\ 1 \le j \le m$$

where $\mathcal{P}_B := \{J \in \mathcal{P} : J \subset B\}$. Evidently $\sum_{j=1}^m \hat{\kappa}^2_{A_j} = 1$. We then write $H_{\mathbf{T}}(\alpha)$ for the corresponding Hamiltonian. From the very construction, for any N, the distance on the configuration space given by $d_{\mathbf{T}}(\alpha, \alpha') = \|H_{\mathbf{T}}(\alpha) - H_{\mathbf{T}}(\alpha')\|_2$ is (deterministically) an ultrametric.

We define then

$$f(\beta, \mathbf{T}) := \lim_{N\to\infty} \frac{1}{N} \mathbb{E} \log Z_{N,\mathbf{T}},$$

where $Z_{N,\mathbf{T}}$ is the partition function corresponding to the above coarse grained GREM model with $\mathcal{P}$ replaced by $\mathbf{T}$ and the corresponding parameters $\hat{\kappa}$.

Theorem 7.1
There exists a nested sequence $\mathbf{T}$ *satisfying*

$$f(\beta) = f(\beta, \mathbf{T})\,,$$

and

$$f(\beta) \le f(\beta, \mathbf{T})$$

holds for any $\mathbf{T}$.

This is proved in [13], and reveals already a rough form of ultrametricity, as it implies that the free energy is the same as that coming from a course graining with a tree structure. "Real" ultrametricity of the Gibbs measure in the form of (7.1) requires an irreducibility assumption which is complicated to state generally. An example where this irreducibility is not satisfied is when $\mathcal{P}$ contains disjoint sets A, B for which there is no chain $C_0 = A, C_1, \ldots, C_{m-1}, C_m = B,\ C_i \in \mathcal{P}$ with $C_i \cap C_{i+1} \neq \emptyset,\ 1 \le i \le m-1$. This is evidently satisfied for instance in the example (7.2).

If the irreducibility assumptions is satisfied, then the limiting Gibbs distribution together with the naturally defined overlaps converges as $N \to \infty$ to a Poisson-Dirichlet point process with overlap distributions given by the coalescent. This is proved in [14], but we will not go into that here.

We will give now explanations of the key points of the proof of the above theorem. A simplifying feature is that one can compute f with a truncated second moment method, leading to a variational formula which at first sight does not "look" ultrametric, but where one sees later, that it in fact is.

To evaluate the free energy, it is convenient to investigate first for $\lambda > 0$

$$\mathcal{N}(\lambda) := \#\{\alpha : H(\alpha) \approx \lambda N\}$$

where we don't specify exactly, what $\approx \lambda N$ means. We can take

$$\mathcal{N}(\lambda) := \#\left\{\alpha : \lambda N - \sqrt{N} \leq H(\alpha) \leq \lambda N + \sqrt{N}\right\}.$$

This grows exponential in N, at least for small λ with

$$\mathcal{N}(\lambda) \approx 2^N \exp\left[-S(\lambda) N\right],$$

and we would get

$$f(\beta) = \sup_{\lambda} \{\lambda\beta - S(\lambda)\} + \log 2,$$

where the supremum is restricted to λ where $\mathcal{N}(\lambda) \neq 0$. In order to determine $\mathcal{N}(\lambda)$, one splits

$$\lambda = \sum_{J \in \mathcal{P}} \lambda_J,$$

and tries to evaluate for $\lambda = \{\lambda_J\}_{J \in \mathcal{S}}$

$$\mathcal{N}(\lambda) := \#\left\{\alpha : X^J_{\alpha_J} \approx \lambda_J N,\ \forall J \in \mathcal{P}\right\},$$

and optimizes over λ. One restriction to have $\mathcal{N}(\lambda) \neq 0$ is certainly

$$\frac{\lambda_J^2}{2\kappa_J^2} \leq \frac{|J|}{K} \log 2,$$

as otherwise, there are simply not enough α_J's. However, this is not sufficient. One needs that for *all* $B \subset [K]$, not just for the ones in $\mathcal{P}$, one has

$$\sum_{J \in \mathcal{P},\ J \subset B} \frac{\lambda_J^2}{2\kappa_J^2} \leq \frac{|B|}{K} \log 2.$$

We write Δ^+ for the set of $\lambda = \{\lambda_J\}_{J \in \mathcal{P}}$ satisfying $\lambda_J \geq 0$ and the above restriction. Somewhat surprisingly, these restrictions are enough in the $N \to \infty$ limit, and one gets

Lemma 7.2

$$f(\beta) = \sup_{\lambda \in \Delta^+} \left\{\sum\nolimits_{J \in \mathcal{P}} \left(\beta\lambda_J - \frac{\lambda_J^2}{2\kappa_J^2}\right)\right\} + \log 2.$$

The proof of this lemma resembles the proof in the REM case.

At first sight, the right hand side does not give any indication of ultrametricity. However, one gets at least upper bounds by relaxing the conditions in Δ^+. For any

nested sequence $\mathbf{T}$ of B's (which not necessarily belong to $\mathcal{P}$) as defined above in (7.3), define

$$\Delta_{\mathbf{T}}^{+} := \left\{ \lambda \in \mathbf{\Delta}^{+} : \sum_{J \in \mathcal{P},\ J \subset B} \frac{\lambda_J^2}{2\kappa_J^2} \leq \frac{|B|}{K} \log 2,\ \forall B \in \mathbf{T} \right\}.$$

Evidently, $\Delta^{+} \subset \Delta_{\mathbf{T}}^{+}$ for all nested sequences, and therefore

$$f(\beta) \leq f(\beta, \mathbf{T}) = \sup_{\lambda \in \Delta_{\mathbf{T}}^{+}} \left\{ \sum\nolimits_{J \in \mathcal{P}} \left(\beta \lambda_J - \frac{\lambda_J^2}{2\kappa_J^2} \right) \right\} + \log 2.$$

The proof of Theorem 7.1 therefore follows from the following result:

Lemma 7.3
There exists a nested sequence $\mathbf{T}$ *such that*

$$\sup_{\lambda \in \Delta^{+}} \left\{ \sum\nolimits_{J \in \mathcal{P}} \left(\beta \lambda_J - \frac{\lambda_J^2}{2\kappa_J^2} \right) \right\} = \sup_{\lambda \in \Delta_{\mathbf{T}}^{+}} \left\{ \sum\nolimits_{J \in \mathcal{P}} \left(\beta \lambda_J - \frac{\lambda_J^2}{2\kappa_J^2} \right) \right\} \tag{7.4}$$

As this lemma is really the crucial fact behind the ultrametricity (at least the one in "weak" form), we give the construction of $\mathbf{T}$.

For $B \subset [K]$ write

$$\phi(B) := \sum_{J \in \mathcal{P},\ J \subset B} \kappa_J^2,$$

and for $B \subset A$

$$\rho(B, A) := \sqrt{\frac{K^{-1}(|A| - |B|)}{\phi(A) - \phi(B)}},$$
$$\hat{\rho}(B) := \min_{A : B \subset A,\ B \neq A} \rho(B, A).$$

We construct recursively an increasing sequence

$$A_0 = \emptyset \subset A_1 \subset \cdots \subset A_M = [K], \tag{7.5}$$

and numbers

$$\beta_0 := 0 < \beta_1 < \beta_2 < \cdots < \beta_M < \infty$$

such that for any $0 \leq k \leq M$ the following properties hold:

C1(k)
$$\beta_j = \hat{\rho}(A_{j-1}),\ j \leq k,$$

C2(k) For all $j \leq k$ and any $A \supset A_{j-1}$ which satisfies $\beta_j = \rho(A_{j-1}, A_j)$, one has $A \subset A_{j-1}$. In other words, A_j is the unique maximal set $\supset A_{j-1}$ achieving the minimum over $\rho(A_{j-1}, A)$.

For $k = 0$, the conditions are void. We now assume that $A_1, \ldots, A_k$ have been constructed on the two conditions **C1(k)** and **C2(k)** are satisfied. If $A_k = [K]$, then the construction is finished. Otherwise, we prove the existence of A_{k+1} such that **C1(k+1)** and **C2(k+1)** are satisfied. We put $\beta_{k+1} := \hat{\rho}(A_k)$. First remark that **C1(k)** and **C2(k)** imply that for any $A \supset A_k,\ A \neq A_k$, one has

$$K^{-1}(|A| - |A_k|) > \phi(A) - \phi(A_k).$$

The construction of A_{k+1} is finished by proving that for any two sets $A, A' \supset A_k$ satisfying

$$\rho(A_k, A) = \rho(A_k, A') = \beta_{k+1}, \tag{7.6}$$

one also has

$$\rho(A_k, A \cup A') = \beta_{k+1}. \tag{7.7}$$

Of course, by the very definition of β_{k+1}, one has $\rho(A_k, A \cup A') \geq \beta_{k+1}$.

The crucial property is

$$\phi(A \cup A') + \phi(A \cap A') \geq \phi(A) + \phi(A'), \tag{7.8}$$

which implies

$$\begin{aligned}
&\frac{|A \cup A'| - |A_k|}{K} - \beta_{k+1}^2 \left(\phi(A \cup A') - \phi(A_k)\right) \\
&\leq \frac{|A| + |A'| - |A \cap A'| - |A_k|}{K} - \beta_{k+1}^2 \left(\phi(A) + \phi(A') - \phi(A \cap A') - \phi(A_k)\right) \\
&= \beta_{k+1}^2 \left(\phi(A \cap A') - \phi(A_k)\right) - \frac{|A \cap A'| - |A_k|}{K} \leq 0,
\end{aligned}$$

the first inequality by (7.8), and the equality by (7.6). The final inequality by the definition of β_{k+1}. We have therefore proved (7.7) which finishes the construction of the nested sequence (7.5).

It is then easy to prove that for $\beta \geq 0$, and with this nested sequence, one has (7.4).

7.2 Infinite overlap structures

We come now back to models of the Sherrington-Kirkpatrick type.

For mean-field models, there is no concept of an infinite Gibbs measure, and therefore, strictly speaking, also no mathematically sound definition of "pure states". However, for SK-type models, there is a way out, which was developed by a number authors, Aizenman, Arguin, Ghirlanda, Guerra, and others, and finally most successfully by Panchenko. The basic observation is that essential all relevant information is encoded in the law of the overlaps of replicas under the $\mathbb{P}$-average of the infinite product of the Gibbs measure. To be precise: Consider an arbitrary random probability measure $\mathcal{G}_{N,\omega}$ on $\Sigma_N = \{-1, 1\}^N$, where we assume that the map $\omega \to \mathcal{G}_{N,\omega}(\sigma)$ is measurable for any

$\sigma \in \Sigma_N$. The infinite product $\mathcal{G}_{N,\omega}^{\otimes\mathbb{N}}$ is a probability measure on $\Sigma_N^{\otimes\mathbb{N}}$, depending still (measurably) on ω. Then we define the averaged measure by

$$\nu(d\sigma) \stackrel{\text{def}}{=} \int \mathcal{G}_{N,\omega}^{\otimes\mathbb{N}}(d\sigma)\,\mathbb{P}(d\omega)$$

on $\Sigma_N^{\otimes\mathbb{N}}$, the latter equipped with the infinite product σ-field.

Next, we consider overlaps from "replicas": Given $\left\{\sigma^\ell\right\}_{\ell\in\mathbb{N}} \in \Sigma_N^{\otimes\mathbb{N}}$, $\sigma^\ell = \left(\sigma_1^\ell, \ldots, \sigma_N^\ell\right)$, we define the infinite matrix R^N of overlaps with components

$$R_{\ell,\ell'}^N \stackrel{\text{def}}{=} \frac{1}{N}\sum_{i=1}^N \sigma_i^\ell \sigma_i^{\ell'}.$$

The matrix elements are in $[-1,1]$, and the diagonal elements are 1. Evidently, the matrix is symmetric and positive semidefinite. We write ρ_N for the distribution of R^N under ν. The set $\mathbb{M}$ of infinite, symmetric, positive semi-definite matrices with entries $\in [-1,1]$ is clearly a compact space under the topology of componentwise convergence. The set of probability measures $\mathcal{P}(\mathbb{M})$ on this space is therefore a compact space, too, under weak convergence. The sequence $\{\rho_N\}$ has therefore convergent subsequences, and one can consider the set of possible limits of subsequences, as $N \to \infty$.

This concept is very ingenious: In classical (short range) systems, one would try to consider limits $\lim_{N\to\infty} \mathcal{G}_N$. This does not make sense for mean field models, as the interactions between different sides vanish in the $N \to \infty$ limit. The above concept of limits $\lim_{N\to\infty} \rho_N$, maybe along subsequences, makes however perfectly sense, and there is a beautiful structure theorem for the possible limits. It turns out that all possible limits are generated in an abstract way similarly as the finite N laws. The limits can in fact be described through a randomization, as will be explained now. This randomization acts as a kind of an "infinite Gibbs measure", but it is not constructed from the original Gibbs measure, but only through the distribution of the overlaps, and an abstract representation theorem.

The distribution ρ_N has an important symmetry property: If $\tau : \mathbb{N} \to \mathbb{N}$ is a permutation of finitely many elements, let $\hat{\tau} : \mathbb{M} \to \mathbb{M}$ be the mapping which exchanges the indices of the matrices accordingly. Evidently ν

$\rho_N \hat{\tau}^{-1} = \rho_N$ where $\rho_N \hat{\tau}^{-1}$ denotes the induced measure under the mapping $\hat{\tau}$. Let $\mathcal{P}^{\text{inv}}(\mathbb{M})$ be the set of probability measures on $\mathbb{M}$ which have this invariance property. $\mathcal{P}^{\text{inv}}(\mathbb{M})$ is a closed subset of $\mathcal{P}(\mathbb{M})$. We formulate now the key abstract representation property for elements of $\mathcal{P}^{\text{inv}}(\mathbb{M})$. For that, let H be one of the standard (real) infinite dimensional separable Hilbert spaces with Borel-σ-field $\mathcal{H}$.

Theorem 7.4 (Dovbysh-Sudakov)
Any $\rho \in \mathcal{P}^{\text{inv}}(\mathbb{M})$ can be represented in the following way. There exist a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and a Markov kernel $\mathcal{G}$ from $(\Omega, \mathcal{F})$ to $(H \times \mathbb{R}^+, \mathcal{H} \otimes \mathcal{B}_{\mathbb{R}^+})$ such that ρ is the law of $(H \times \mathbb{R}^+)^{\mathbb{N}} \ni ((h_\ell, a_\ell))_{\ell\in\mathbb{N}} \to \left(\langle h_\ell, h_{\ell'}\rangle\left(1-\delta_{\ell,\ell'}\right) + a_\ell \delta_{\ell,\ell'}\right)_{\ell,\ell'} \in \mathbb{M}$ under $\int \mathbb{P}(d\omega)\,\mathcal{G}^{\otimes\mathbb{N}}(\omega,\cdot)$. ($\langle\cdot,\cdot\rangle$ is the inner product in H.)

Essentially, the Dovbysh-Sudakov theorem says that any measure in $\mathcal{P}^{\text{inv}}(\mathbb{M})$ is the law of the matrix of inner products under the averaged infinite product of a random probability on H. There is a slight modification on the diagonal, as this applies only to the off-diagonal part of the random matrix, and the diagonal elements are produced by a separate random mechanism, as described above.

For the case of the overlap distribution ρ_N, the representation is already given by the Gibbs distribution, as we can regard Σ_N as a subset of a Hilbert space. The theorem therefore constructs a kind of a substitute for the limit of the Gibbs measures.

7.3 The Ghirlanda-Guerra identities imply ultrametricity

The chapter here is based on Panchenko's book [38] and the overview in [17].

As remarked in the last section, we consider a Markov kernel $\mathcal{G}$ from a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ to the unit ball of a separable real Hilbert space H. This means that $\mathcal{G}(\omega, \cdot)$ is for every $\omega \in \Omega$ a probability measure on the unit ball of the Hilbert space, and there is a measurable dependence on ω: Consider then ν to be the measure $\nu := \int \mathcal{G}(\omega, \cdot)^{\otimes \mathbb{N}} \mathbb{P}(d\omega)$ on $H^{\mathbb{N}}$.

It will occasionally be necessary to integrate functions ϕ defined on $\Omega \times H^{\mathbb{N}}$:

$$\int \left[\int \phi(\omega, \sigma) \mathcal{G}^{\otimes \mathbb{N}}(\omega, d\sigma) \right] \mathbb{P}(d\omega).$$

By an abuse of notation, we also write $\int \phi d\nu$ for that. For $\sigma = \left(\sigma^1, \sigma^2, \ldots\right) \in H^{\mathbb{N}}$, define the overlap $R_{ij} \overset{\text{def}}{=} \left\langle \sigma^i, \sigma^j \right\rangle$. $R = (R_{ij})$ is a random symmetric matrix which is positive semidefinite. From the very definition of ν, one sees that the distribution of (R_{ij}) is the same as that of $\left(R_{\pi(i),\pi(j)}\right)$ for any permutation π of finitely many elements.

Definition 7.5
We say that the pair $(\mathbb{P}, \mathcal{G})$ satisfies the Ghirlanda–Guerra identity, if

$$\nu\left(R_{1,N+1} \in A \middle| (R_{ij})_{i,j \le N}\right) = \frac{1}{N} \sum_{j=2}^{N} 1_A(R_{1,j}) + \frac{1}{N} \nu(R_{12} \in A). \tag{7.9}$$

The main result of Panchenko is

Theorem 7.6
Assume (7.9). Then

$$\nu(R_{1,2} \ge \min(R_{1,3}, R_{2,3})) = 1. \tag{7.10}$$

Theorem 7.7
Assume (7.9). Then the distribution of R under ν is completely characterized by the Parisi measure

$$\zeta := \nu(R_{1,2} \in \cdot).$$

In the case where ζ is supported by a finite set, the ν-law of R is that coming from the Ruelle cascades with this Parisi measure.

Remark 7.8

a) That ultrametricity, Ghirlanda-Guerra and a finite support imply that the distribution of R is that of a Ruelle cascade is not very difficult, and was well known before. The really important step was the proof of the ultrametricity.

b) Using the coalescent process introduced in Section 4.3, the Ruelle overlap model can directly be defined with an arbitrary Parisi measure, and without taking recourse to a finite support of ζ. There are however many steps, also in the second part of [10], which use an approximation procedure by finite stage cascades. It would certainly be valuable to have this point cleared up, and have proofs avoiding this approximation procedure.

There are a number of relatively easy properties one can draw from the Ghirlanda-Guerra identities. For a proof of the following result, see [38], Theorem 2.15 and 2.16:

Proposition 7.9
If $(\mathbb{P}, G)$ satisfies the Ghirlanda-Guerra identities, then

a) (Talagrand's positivity result)

$$\nu\left(R_{1,2} \geq 0\right) = 1.$$

b) There exists a constant $q^* \leq 1$ such that

$$\nu\left(\left\|\sigma^i\right\|_H^2 = q^*\right) = 1.$$

b) says that the self-overlaps are all constant. The matrix

$$R^{(3)} \stackrel{\text{def}}{=} \begin{pmatrix} R_{1,1} & R_{1,2} & R_{1,3} \\ R_{2,1} & R_{2,2} & R_{2,3} \\ R_{3,1} & R_{3,2} & R_{3,3} \end{pmatrix}$$

therefore takes values in the set of symmetric positive semidefinite matrices with q^* on the diagonal. Denote by $\mathbf{S}^{(n)}$ the compact support of the distribution of $(R_{ij})_{i,j\leq n}$ under ν. The ultrametricity claim (7.10) is equivalent with the statement that if

$$\begin{pmatrix} q^* & a & b \\ a & q^* & c \\ b & c & q^* \end{pmatrix}$$

are in the support, then none of a, b, c is strictly smaller than the other two (among a, b, c). Without loss of generality, we may assume that $a \leq b \leq c$. So, by a slight abuse of notation, we write $\mathbf{S}^{(3)}$ for the set of such triples (a, b, c) such that the above matrix is in the support. Ultrametricity is violated if we find an $(a, b, c) \in \mathbf{S}^{(3)}$ with $a < b$. One easily sees that one needs only to consider $c < q^*$: If $(a, b, q^*) \in \mathbf{S}^{(3)}$ then for any $\varepsilon > 0$, the ν-probability to find $\sigma^{(2)}, \sigma^{(3)}$ with $\left\langle \sigma^{(2)}, \sigma^{(3)} \right\rangle \geq q^* - \varepsilon$ is positive. But this implies

(because $\left\|\sigma^{(1)}\right\|^2 = \left\|\sigma^{(2)}\right\|^2 = q^*$), that $\sigma^{(2)}, \sigma^{(3)}$ can be chosen arbitrarily close. This easily implies that (a, b, q^*) with $a < b$ cannot be in $\mathbf{S}^{(3)}$. Therefore, the difficult task remains to prove

$$a < b \leq c < q^* \Longrightarrow (a, b, c) \notin \mathbf{S}^{(3)}. \tag{7.11}$$

Panchenko's proof relies an the following replication property

Proposition 7.10
Assume that $(a, b, c) \in \mathbf{S}^{(3)}$, and that the Ghirlanda-Guerra identities are satisfied. Then for every m, there exists a $3m \times 3m$ matrix $(r_{ij})_{i,j \leq 3m} \in \mathbf{S}^{(3m)}$ which satisfies with $I_k := \{(k-1)\, m + 1, \ldots, km\}$, $k = 1, 2, 3$.

- $$r_{ij} \leq c, \ \forall i \neq j \tag{7.12}$$

- $$r_{ij} = \begin{cases} a & \text{for } i \in I_1, \ j \in I_2 \\ b & \text{for } i \in I_1, \ j \in I_3 \\ c & \text{for } i \in I_2, \ j \in I_3 \end{cases} \tag{7.13}$$

It is not difficult to see that the above proposition implies $a = b$ if $a \leq b \leq c < q^*$: Put

$$\bar{\sigma}^i := \frac{1}{m} \sum_{j \in I_k} \sigma^j$$

If $\sigma^i \in H$ satisfy $\left\langle \sigma^i, \sigma^j \right\rangle = r_{ij}$ and the r_{ij} satisfy (7.13), then

$$\left\langle \bar{\sigma}^1, \bar{\sigma}^2 \right\rangle = a, \ \left\langle \bar{\sigma}^1, \bar{\sigma}^3 \right\rangle = b, \ \left\langle \bar{\sigma}^2, \bar{\sigma}^3 \right\rangle = c.$$

Furthermore

$$\left\| \bar{\sigma}^k \right\|_H^2 = \frac{q^*}{m} + \frac{1}{m^2} \sum_{i \neq j, \ \in I_k} \left\langle \sigma^i, \sigma^j \right\rangle \leq \frac{q^* - c}{m} + c.$$

Therefore,

$$\begin{aligned} \left\| \bar{\sigma}^2 - \bar{\sigma}^3 \right\|_H^2 &= \left\| \bar{\sigma}^2 \right\|_H^2 + \left\| \bar{\sigma}^3 \right\|_H^2 - 2 \left\langle \bar{\sigma}^2, \bar{\sigma}^3 \right\rangle \\ &\leq 2 \frac{q^* - c}{m} + 2c - 2c = \frac{2\,(q^* - c)}{m}, \end{aligned}$$

On the other hand

$$\begin{aligned} b - a &= \left\langle \bar{\sigma}^1, \bar{\sigma}^3 \right\rangle - \left\langle \bar{\sigma}^1, \bar{\sigma}^2 \right\rangle \\ &\leq \left\| \bar{\sigma}^1 \right\|_H \left\| \bar{\sigma}^2 - \bar{\sigma}^3 \right\|_H \leq \frac{2\,(q^* - c)}{m} \end{aligned}$$

for any m, implying $b = a$.

Therefore, in order to prove ultrametricity under Ghirlanda-Guerra, the crucial step is to prove the replication property of Proposition 7.10.

The basis of the proof of this proposition is a proof that the Ghirlanda-Guerra identities imply a sophisticated invariance property of ν. Let $f : \mathbb{R} \to \mathbb{R}$ be bounded and measurable and define $F_f : \Omega \times H \to \mathbb{R}$ by

$$F_f(\omega, \sigma) := \int \exp\left[f\left(\left\langle \sigma, \sigma' \right\rangle\right)\right] \mathcal{G}\left(\omega, d\sigma'\right). \tag{7.14}$$

Proposition 7.11
Let $N \in \mathbb{N}$ and Φ be a measurable real valued function on the space of $n \times n$-matrices with entries bounded by 1, and let f, F be as above. If the Ghirlanda-Guerra identities are satisfied, then

$$\nu(\Phi) = \nu\left(\Phi \frac{\exp\left[\sum_{i=1}^{n-1} f(R_{i,n}) + \nu(f(R_{1,2}))\right]}{F_f(\sigma^n)^n}\right).$$

Here as usual $R_{ik} = \left\langle \sigma^i, \sigma^k \right\rangle$.

Remark 7.12
We are stressing here a bit the notation. The function which is integrated on the right hand side is not a function of σ alone as F_f also depends explicitly on ω. So the integral on the right hand side is an integral with respect to the semi-direct product measure $\mathbb{P}(d\omega)\,\mathcal{G}_\omega^{(n)}(d\sigma)$. The somewhat puzzling point is that such an equation can be a consequence of the Ghirlanda-Guerra identities which refer only to $\int \mathcal{G}_\omega^{(n)} \mathbb{P}(d\omega)$.

Proof. It seems to be difficult to get the identity directly out from Ghirlanda-Guerra. The trick is to replace f by tf, $t \in [0,1]$, and compute derivatives. It turns out that of the expression on the right hand side, arbitrary derivatives at $t = 0$ can then be expressed in terms of ν expectations, and are all 0 by Ghirlanda-Guerra. Using analyticity in t, on the obtains that as a function of t, the expression is constant, at least in some interval containing 0. Using some additional bounds, one obtains that it is constant for all $t \in [0,1]$. For the details, see the monograph by Panchenko [38].

The computation of the derivatives is a bit messy but not difficult. I do it for the first derivative. The reader will have no difficulty to do the higher orders. Put

$$\varphi(t) := \nu\left(\Phi \frac{\exp\left[t\sum_{i=1}^{n-1} f(R_{i,n}) + t\nu(f(R_{1,2}))\right]}{F_{tf}(\sigma^n)^n}\right).$$

$$\frac{d\varphi(t)}{dt} = \nu\left(\Phi\left(R^{(n)}\right)\left[\sum_{i=1}^{n-1} f(R_{i,n}) + \nu(f(R_{1,2}))\right]\frac{\exp\left[t\sum_{i=1}^{n-1} f(R_{i,n}) + t\nu(f(R_{1,2}))\right]}{F_{tf}(\sigma^n)^n}\right)$$
$$- n\nu\left(\Phi\left(R^{(n)}\right)\frac{\exp\left[t\sum_{i=1}^{n-1} f(R_{i,n}) + t\nu(f(R_{1,2}))\right]}{F_{tf}(\sigma^n)^{n+1}}\frac{dF_{tf}(\sigma^n)}{dt}\right),$$

$$\frac{dF_{tf}\left(\sigma^n\right)}{dt} = \int f\left(\left\langle \sigma^n, \sigma' \right\rangle\right) \exp\left[tf\left(\left\langle \sigma, \sigma' \right\rangle\right)\right] \mathcal{G}\left(\omega, d\sigma'\right).$$

This leads to

$$\begin{aligned}\left.\frac{d\varphi\left(t\right)}{dt}\right|_{t=0} &= \nu\left(\Phi\left(R^{(n)}\right)\left(\sum_{i=1}^{n-1} f\left(R_{i,n}\right) + \nu\left(f\left(R_{1,2}\right)\right)\right)\right) \\ &\quad - n\nu\left(\Phi\left(R^{(n)}\right) f\left(\left\langle \sigma^n, \sigma^{n+1} \right\rangle\right)\right),\end{aligned}$$

which is 0 by Ghirlanda-Guerra. ■

Corollary 7.13
Let $w : \mathbb{R} \to \mathbb{R}$ be a bounded measurable function, and f, F, Φ as above. Define

$$\begin{aligned}S_w\left(\omega, \sigma\right) &:= \int w\left(\left\langle \sigma, \sigma' \right\rangle\right) \mathcal{G}_\omega\left(d\sigma\right), \\ T_w\left(\omega, \sigma\right) &:= F_f\left(\omega, \sigma\right)^{-1} \int w\left(\left\langle \sigma, \sigma' \right\rangle\right) \mathrm{e}^{f\left(\left\langle \sigma, \sigma' \right\rangle\right)} \mathcal{G}_\omega\left(d\sigma\right).\end{aligned}$$

Then, for a measurable function $\psi : \mathbb{R} \to \mathbb{R}$

$$\begin{aligned}&\nu\left(\Phi\left(R^{(n)}\right)\psi\left(S\left(\sigma^n\right)\right)\right) \\ &= \nu\left(\Phi\left(R^{(n)}\right)\psi\left(T\left(\sigma^n\right)\right)\frac{\exp\left[\sum_{i=1}^{n-1} f\left(R_{i,n}\right) + \nu\left(f\left(R_{1,2}\right)\right)\right]}{F\left(\sigma^n\right)^n}\right).\end{aligned}$$

Proof. It suffices to prove the statement for polynomials ψ. So, we take $\psi\left(x\right) = x^k$.

$$\begin{aligned}\Phi\left(R^{(n)}\right) T\left(\sigma^n\right)^k = \frac{\Phi\left(R^{(n)}\right)}{F\left(\sigma^n\right)^k} \int \exp\left[\sum\nolimits_{j=1}^{k} f\left(R_{n,n+j}\right)\right] \\ \times \prod_{j=1}^{k} w\left(R_{n,n+j}\right) G\left(\cdot, d\sigma^{n+j}\right).\end{aligned}$$

Therefore, if we put

$$\Phi'\left(R^{(n+k)}\right) \stackrel{\text{def}}{=} \Phi\left(R^{(n)}\right) \prod_{j=1}^{k} w\left(R_{n,n+j}\right),$$

the right hand side of the claimed equation is

$$\nu\left(\Phi'\left(R^{(n+k)}\right)\frac{\exp\left[\sum_{i=1}^{n-1} f\left(R_{i,n}\right) + \sum_{i=n+1}^{n+k} f\left(R_{i,n}\right) + \nu\left(f\left(R_{1,2}\right)\right)\right]}{F\left(\sigma^n\right)^{n+k}}\right)$$

which, by the previous proposition, equals

$$\nu\left(\Phi\left(R^{(n)}\right)\prod_{j=1}^{k}w\left(R_{n,n+j}\right)\right)=\nu\left(\Phi\left(R^{(n)}\right)S\left(\sigma^{n}\right)^{k}\right).$$

■

Proof of Proposition 7.10. The replication property follows by induction of the following statement:

Claim 7.14
Let $A\in\mathbf{S}^{(n)}$ satisfy $a_n^*\overset{\text{def}}{=}\max\left(a_{1,n},\ldots,a_{n-1,n}\right)<q^*$. Then there exists an extension $A'\in\mathbf{S}^{(n+1)}$ of A such that $a_{i,n+1}=a_{i,n}$ for $i\leq n-1$, and $a_{n,n+1}\leq a_n^*$. (Extension here means that A is the $n\times n$-matrix obtained from removing the last column and row of A'.)

To prove the claim, we will prove for all $\varepsilon>0$, one has

$$\nu\left(R^{(n)}\in U_{\varepsilon}\left(A\right),\ \left|R_{i,n+1}-a_{i,n}\right|\leq\varepsilon,\ i\leq n-1,\ R_{n,n+1}<a_n^*+\varepsilon\right)>0, \tag{7.15}$$

where $U_{\varepsilon}\left(A\right)$ denotes the componentwise ε-neighborhood of A, which proves the claim. The argument is best done indirectly, by assuming that for some $\varepsilon>0$

$$\nu\left(R^{(n)}\in U_{\varepsilon}\left(A\right),\ \left|R_{i,n+1}-a_{i,n}\right|\leq\varepsilon,\ i\leq n-1,\ R_{n,n+1}<a_n^*+\varepsilon\right)=0 \tag{7.16}$$

Define

$$\Sigma\left(\sigma^{(n-1)}\right)\overset{\text{def}}{=}\left\{\sigma\in H:\left|\left\langle\sigma,\sigma^{i}\right\rangle-a_{i,n}\right|\leq\varepsilon,\ i\leq n-1\right\},$$

and let A' be the matrix obtained from A by erasing the last row and the last column. A reformulation of (7.16) gives

$$\nu\left(R^{(n-1)}\in U_{\varepsilon}\left(A'\right),\ \sigma^{n},\sigma^{n+1}\in\Sigma\left(\sigma^{(n-1)}\right),\ \left\langle\sigma^{n},\sigma^{n+1}\right\rangle<a_n^*+\varepsilon\right)=0.$$

We use Corollary 7.13 with $w\left(x\right)=1_{x\geq a_n^*+\varepsilon}$, and $f=tw$ with $t\geq0$ which will be chosen later. Then

$$S_w\left(\omega,\sigma^{n}\right)\overset{\text{def}}{=}G\left(\omega,\left\{\sigma:\left\langle\sigma,\sigma^{n}\right\rangle\geq a_n^*+\varepsilon\right\}\right).$$

$A\in\mathbf{S}^{(n)}$ implies that given $\varepsilon>0$, there exist $\delta>0$, $0<p<1/2$, such that

$$\nu\left(R^{(n)}\in U_{\varepsilon}\left(A\right),\ p\leq S\left(\sigma^{n}\right)\leq1-p\right)\geq\delta. \tag{7.17}$$

F according to (7.14) is:

$$F\left(\omega,\sigma^{n}\right)=S\left(\omega,\sigma^{n}\right)\left(\mathrm{e}^{t}-1\right)+1\geq1.$$

By Corollary 4.2, applied to $\psi(x) = 1_{p \le x \le 1-p}$, we get

$$\nu\left(\frac{\mathrm{e}^{\sum_{i=1}^{n-1} f(R_{i,n}) + \nu(f(R_{1,2}))}}{F(\sigma^n)^n}; R^{(n)} \in U_\varepsilon(A),\ p \le T(\sigma^n) \le 1-p\right) \ge \delta.$$

Evidently, $f(R_{i,n}) = 0$ on $R^{(n)} \in U_\varepsilon(A)$, and

$$\nu(f(R_{1,2})) = t\nu(R_{1,2} \ge a_n^* + \varepsilon) = t\gamma,$$

where $\gamma < 1$. Therefore, as $F(\sigma^n) \ge 1$, we get

$$\nu\left(R^{(n)} \in U_\varepsilon(A),\ T(\sigma^n) \le 1-p\right) \ge \delta \mathrm{e}^{-\gamma t}. \tag{7.18}$$

An elementary computation gives that $T(\omega, \sigma^n) \le 1-p$ implies

$$S(\omega, \sigma^n) \le \frac{1-p}{p}\mathrm{e}^{-t}.$$

Consider now

$$\Lambda\left(\omega, \sigma^{(n-1)}\right) \stackrel{\text{def}}{=} \left\{\sigma^n \in H : \sigma^n \in \Sigma\left(\sigma^{(n-1)}\right),\ T(\omega, \sigma^n) \le 1-p\right\},$$

and

$$\Pi \stackrel{\text{def}}{=} \left\{\left(\omega, \sigma^{(n-1)}\right) \in \Omega \times H^{n-1} : G\left(\omega, \Lambda\left(\omega, \sigma^{(n-1)}\right)\right) > 0\right\}$$

Then

$$\begin{aligned}
&\nu\left(R^{(n)} \in U_\varepsilon(A),\ T(\sigma^n) \le 1-p\right) \\
&= \int \mathbb{P}(d\omega) \int_{U_\varepsilon(A')} G^{\otimes(n-1)}\left(\omega, d\sigma^{(n-1)}\right) G\left(\omega, \Lambda\left(\omega, \sigma^{(n-1)}\right)\right).
\end{aligned}$$

For any $\left(\omega, \sigma^{(n-1)}\right) \in \Pi$, we can choose a $\sigma' \in \Lambda\left(\omega, \sigma^{(n-1)}\right)$ for which one has

$$T(\omega, \sigma') \le 1-p \Longrightarrow S(\omega, \sigma') \le \frac{1-p}{p}\mathrm{e}^{-t}.$$

However, fixing this σ', almost all $\sigma^n \in \Sigma\left(\sigma^{(n-1)}\right)$ satisfy $\langle \sigma', \sigma^n \rangle \ge a_n^* + \varepsilon$ by (7.16), and therefore, for any $\left(\omega, \sigma^{(n-1)}\right) \in \Pi$, one has

$$G\left(\omega, \Lambda\left(\omega, \sigma^{(n-1)}\right)\right) \le G\left(\omega, \Sigma\left(\sigma^{(n-1)}\right)\right) \le S(\omega, \sigma') \le \frac{1-p}{p}\mathrm{e}^{-t}.$$

Therefore

$$\begin{aligned}
&\nu\left(R^{(n)} \in U_\varepsilon(A),\ T(\sigma^n) \le 1-p\right) \\
&\le \int \left(\mathbb{P} \otimes G^{\otimes(n-1)}\right)\left(d\left(\omega, \sigma^{(n-1)}\right)\right) 1_\Pi\left(\left(\omega, \sigma^{(n-1)}\right)\right) G\left(\omega, \Lambda\left(\omega, \sigma^{(n-1)}\right)\right) \\
&\le \frac{1-p}{p}\mathrm{e}^{-t}.
\end{aligned}$$

As the left hand side is $\geq \delta \mathrm{e}^{-\gamma t}$, by (7.18), it follows that

$$\delta \leq \frac{1-p}{p} \mathrm{e}^{\gamma t} \mathrm{e}^{-t},$$

which leads to a contradiction when choosing t large enough, as $\gamma < 1$. ■

8 Ultrametricity implies the Parisi formula

Panchenko's proof of ultrametricity sketched in the last section gives a proof of ultrametricity for the SK- and related models, not quite in the form given in (7.1), but for a model with a slightly perturbed Hamiltonian. The reason is that the Ghirlanda-Guerra identities can be proved only for the SK-model with a perturbed Hamiltonian. We give a rough outline of the chain of arguments. Unfortunately, it is still technically quite involved.

The argument is based on (5.3) with $M = 1$ and the η_σ given by the SK Gibbs distribution $\mathcal{G}'_N$ with slightly changed temperature parameter which gives

$$\begin{aligned} A_N &:= \mathbb{E} \log Z_{N+1} - \mathbb{E} \log Z_N \\ &= \mathbb{E} \log \sum\nolimits_{\sigma \in \Sigma_N,\ \tau \in \Sigma_1} \mathcal{G}'_N(\sigma) \exp\left[(\beta y_{\sigma,i} + h)\tau\right] \\ &- \mathbb{E} \log \sum\nolimits_{\sigma \in \Sigma_N} \mathcal{G}'_N(\sigma) \exp\left[\beta \sqrt{1/2} \kappa_\sigma\right] \end{aligned} \tag{8.1}$$

Remark, that we can represent κ_σ as

$$\kappa_\sigma = \kappa_{N,\sigma:} = \frac{1}{\sqrt{N(N+1)}} \sum_{1 \leq i < j \leq N} g'_{ij} \sigma_i \sigma_j$$

with new independent g'_{ij}. Therefore,

$$A_N = \mathbb{E} \log \mathcal{E}'_N \cosh(\beta y_N) - \mathbb{E} \log \mathcal{E}'_N \exp[\beta y_N] + \log 2,$$

and

$$\frac{1}{N} \mathbb{E} \log Z_N = \frac{1}{N} \sum_{j=0}^{N-1} A_j$$

leading to

$$f(\beta) = \lim_{N \to \infty} \frac{1}{N} \mathbb{E} \log Z_N \geq \liminf_{N \to \infty} A_N. \tag{8.2}$$

This is the basis for getting a lower bound by relating (8.1) to the Parisi expression. Of course, if one could prove the existence of $\lim_{N \to \infty} A_N$, and identify it with the Parisi expression, one would have finished the proof of the Parisi formula. This seems not to be possible. However, the upper bound for f is known by the Guerra's interpolation technique.

The task which remains is to prove that the right-hand side of (8.2) is bounded from below by the Parisi expression. This is based on the fact that a slightly perturbed SK-model $\mathcal{G}_N^{\text{pert}}$ satisfies the Ghirlanda-Guerra identities as explained in Section 6.

It then follows that any weak limit of the distribution under $\nu^{\text{pert}} \overset{\text{def}}{=} \int G_{N,\beta}^{\text{pert}\otimes\mathbb{N}} d\mathbb{P}$ of the overlaps, where $\mathbb{P}$ governs the laws of the Gaussian interaction variables, along a subsequence,

$$R_{i,j}^N \overset{\text{def}}{=} \frac{1}{N}\sum_{k=1}^{N} \sigma_k^i \sigma_k^j,$$

has the exchangeability property, which, by the Dovbysh-Sudakov theorem, implies the representation as the law of the inner products in a Hilbert space H, under the law given by a random measure G^{DS} on H, with an abstract probability space behind, which we denoted by $\left(\Omega, \mathcal{F}, \mathbb{P}^{\text{DS}}\right)$. The fact that R^N under ν^{pert} satisfies approximative form of Ghirlanda-Guerra implies that $\left(G^{\text{DS}}, \mathbb{P}^{\text{DS}}\right)$ satisfies Ghirlanda-Guerra exactly, and therefore, by the Theorems 7.6 and 7.7, is ultrametric, and determined by the Parisi measure $\zeta\left(dx\right) = \nu^{\text{DS}}\left(R_{1,2} \in dx\right)$. In case ζ has finite support, the distribution is that coming from the Ruelle cascade with this Parisi measure. As we don't know anything about convergence along the original sequence, there can be arbitrarily many of such Dovbysh-Sudakov pairs, and Parisi measures.

It's now tempting to apply this to the evaluation of A_N for large N. We first choose a subsequence, such that $A_{N_k} \to^{k\to\infty} \liminf_{N\to\infty} A_N$. Then we choose a further subsequence of $\{A_{N_k}\}$ such that the law of R^N along this subsequence converges to the law of the inner products under ν^{DS}. We would like to conclude that $\liminf_{N\to\infty} A_N$ can be expressed through ν^{DS}. This is not quite obvious as the A_N cannot directly be expressed through the overlaps. It requires an additional approximation argument. The basic observation is that the covariances of $y_N\left(\sigma\right)$, and $\kappa_N\left(\sigma\right)$ are expressed in terms of the overlaps of σ. This implies that A_N can be approximated by functions of finite restrictions of the overlap matrix. In case the Parisi measure ν has finite support, one can conclude in this way, using Lemma 5.7 that

$$\liminf_{N\to\infty} A_N = \mathcal{P}\left(\zeta\right) \geq \inf_\xi \mathcal{P}\left(\xi\right).$$

In the case of a general Parisi measure, appearing through the limits along the subsequences defined above, an additional approximation is needed, but one concludes that

$$\liminf_{N\to\infty} A_N \geq \inf_\xi \mathcal{P}\left(\xi\right).$$

Together with Guerra's upper bound, this finally proves the Parisi formula.

9 Appendix: The Curie-Weiss model

Given a finite set Σ and a function $H : \Sigma \to \mathbb{R}$, non-random for the moment, i.e. $H\left(\sigma\right)$ is just a number not a random variable, then the Gibbs measure on Σ with inverse

temperature $\beta > 0$ is defined by

$$\mathcal{G}_\beta(\sigma) := \frac{\exp[\beta H(\sigma)]}{Z_\beta},$$

with the **partition function**

$$Z_\beta := \sum_\sigma \exp[\beta H(\sigma)].$$

As remarked before, the physicists would write $\exp[-\beta H(\sigma)]$, but that would mean that all the time we would have to put a minus sign before our Hamiltonians to compensate for that, so I don't do it.

One of the simplest non-trivial mean-field models in classical statistical mechanics is the **Curie-Weiss model** which has a *non-random* Hamiltonian, defined on $\Sigma_N := \{-1,1\}^N$ by

$$H_N(\sigma) := \frac{1}{2N}\sum_{i,j=1}^N \sigma_i\sigma_j = \frac{1}{2N}\left(\sum_{i=1}^N \sigma_i\right)^2,\ \sigma = (\sigma_1,\ldots,\sigma_N)$$

The diagonal term $\sum_{i=j}$ is just 1, and this cancels out with the normalization and does not influence the Gibbs measure. We could therefore as well just take the sum $\sum_{i\neq j}$ which is often done. The key point is that this Hamiltonian reflects an interaction of any individual spin σ_i with the average of the other spins $\Sigma_{j:j\neq i}\sigma_j/(N-1)$. The total "interaction energy" is then

$$\frac{1}{2}\sum_i \sigma_i \frac{\Sigma_{j:j\neq i}\sigma_j}{N-1} = \frac{1}{2}\frac{1}{N-1}\sum_{i\neq j}\sigma_i\sigma_j.$$

That there is $N-1$ instead of N is of no importance for large N.

Occasionally, one also has a so-called external field which give the σ_i a global tilt. Then the Hamiltonian is

$$H_N(\sigma) := \frac{1}{2N}\sum_{i,j=1}^N \sigma_i\sigma_j + h\sum_{i=1}^N \sigma_i.$$

$h \in \mathbb{R}$ is an additional parameter. The Curie-Weiss Gibbs measure is defined by

$$\mathcal{G}_{\beta,h,N}(\sigma) := \frac{1}{Z_{\beta,h,N}}\exp\left[\frac{\beta}{2N}\sum_{i,j=1}^N \sigma_i\sigma_j + \beta h\sum_{i=1}^N \sigma_i\right],$$

where

$$Z_{\beta,h,N} := \sum_{\sigma\in\Sigma_N}\exp\left[\frac{\beta}{2N}\sum_{i,j=1}^N \sigma_i\sigma_j + \beta h\sum_{i=1}^N \sigma_i\right]. \tag{9.1}$$

This model can easily be analyzed by Stirling's formula. The point is that the Hamiltonian is a function of $\bar{\sigma}_N := \frac{1}{N}\sum_{i=1}^N \sigma_i$:

$$H_N(\sigma) = N\left(\frac{1}{2}\bar{\sigma}_N^2 + h\bar{\sigma}_N\right),$$

and Z can be written as expectation under standard coin tossing:

$$Z_N = 2^N E_N^{\mathrm{CT}} \exp\left[N\beta\left(\bar{\sigma}_N^2/2 + h\bar{\sigma}_N\right)\right].$$

The coin tossing expectation E^{CT} can be computed in terms of Stirling's formula up to any precision one likes. The rough large deviation behavior, in the usual large deviation jargon, is

$$P_N^{\mathrm{CT}}(\bar{\sigma}_N \sim x) \sim \exp\left[-NI(x)\right]$$

with the entropy function

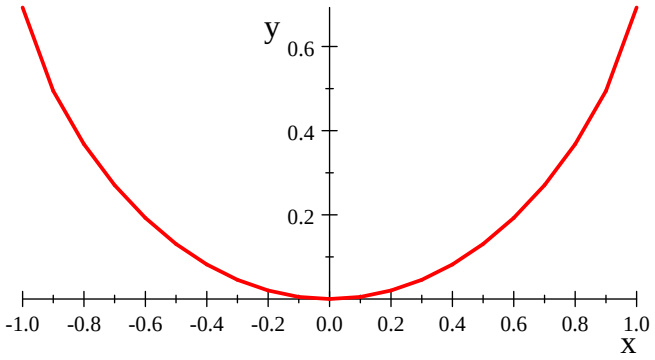
$$I(x) = \begin{cases} \frac{1+x}{2}\log(1+x) + \frac{1-x}{2}\log(1-x) & \text{if } x \in [-1,1] \\ \infty & \text{if } x \notin [-1,1] \end{cases}. \tag{9.2}$$

Therefore,

$$f(\beta,h) = \lim_{N\to\infty}\frac{1}{N}\log Z_N = \log 2 + \sup_x\left[\frac{\beta}{2}x^2 + \beta h x - I(x)\right].$$

For those who are not familiar with these type of arguments, I leave it as an exercise to derive it from Stirling's formula.

The function $x \to I(x) = \frac{1+x}{2}\log(1+x) + \frac{1-x}{2}\log(1-x)$ looks as follows

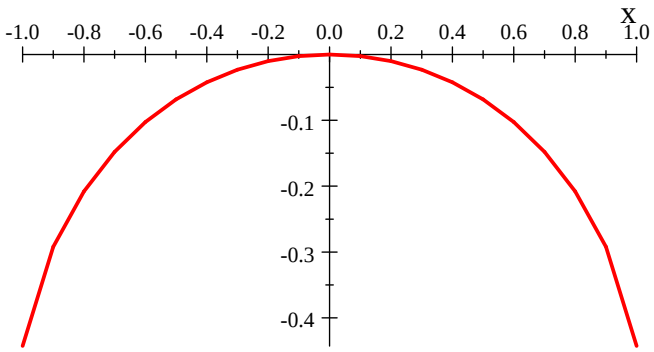


It is of course even. Furthermore, the function is continuous on the full interval $[-1,1]$ with

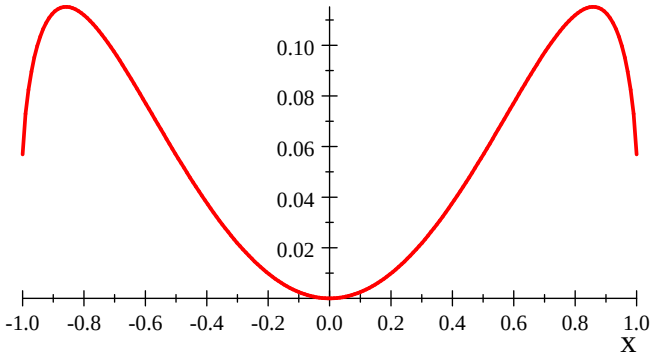
$$\lim_{x\to\pm 1} I(x) = \log 2,$$

but the tangent diverges as $x \to \pm 1$. The behavior of the Curie-Weiss model is determined by the function $x \to g_{\beta,h}(x) := \frac{\beta}{2}x^2 + \beta h x - I(x)$. This depends heavily on β and h.

Case $h = 0$: In this case the above function is even, but there is a crucial difference depending on whether $\beta \le 1$ or $\beta > 1$. Below there are plots for $\beta = 1/2$, and $\beta = 3/2$.



$g_{\beta,h}$ for $h = 0, \beta = 1/2$



$g_{\beta,h}$ for $h = 0, \beta = 3/2$

The crucial difference is coming from the second derivative:

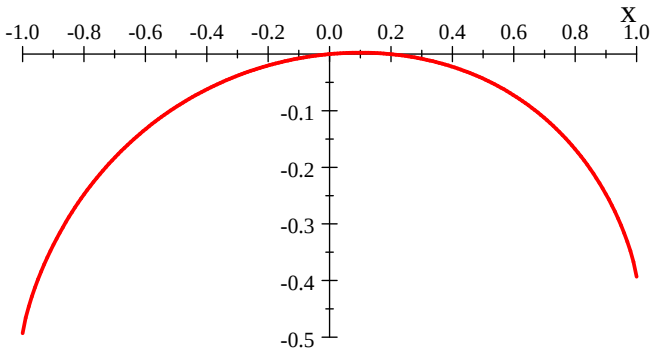
$$\frac{dg}{dx} = \beta x - \left(\frac{1}{2}\log(1+x) - \frac{1}{2}\log(1-x)\right),$$

$$\frac{d^2g}{dx^2} = \beta - \frac{1}{2(1+x)} - \frac{1}{2(1-x)}.$$

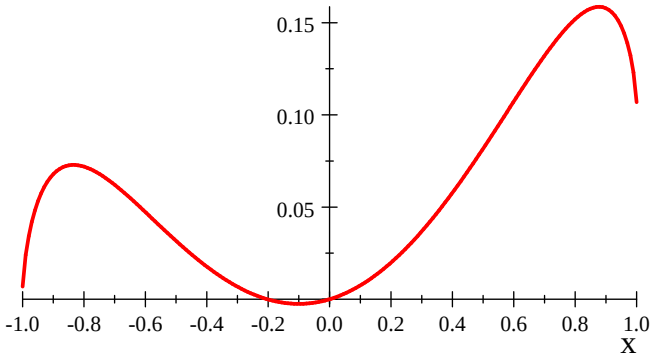
For $\beta < 1$, the second derivative is negative everywhere, and therefore the function is strictly concave with a unique maximum at 0. This remains true for $\beta = 1$, where the second derivative is 0 at 0, but negative for $x \neq 0$. However, for $\beta > 1$, the second derivative is positive at 0, and negative for x sufficiently close to ± 1. Therefore, 0 is a local minimum, and the maxima of the function are elsewhere. Setting the first derivative 0, one gets the equation $m = \tanh(\beta m)$ for the maximum m. For $\beta \leq 1$, there is just the solution 0 for this equation, but for $\beta > 1$, there are 2 other solutions $\pm m_\beta$.

The free energy $f(\beta, 0)$ is $\log 2$ for $\beta \leq 1$, and starts to increase for $\beta > 1$.

Case $h \neq 0$: In that case, 0 is never a maximum. The curve still depends on the value of β, but it has always one unique global maximum, although it may have a local maximum besides that. Below are two examples, both with $h = 1/20$, and the first with $\beta = 1/2$, while the second with $\beta = 3/2$:



$g_{\beta,h}$ for $h = 1/20$, $\beta = 1/2$



$g_{\beta,h}$ for $h = 1/20$, $\beta = 3/2$

It is easily checked that there is a unique $m_{\beta,h} \neq 0, \pm 1$, such that

$$g_{\beta,h}\left(m_{\beta,h}\right) = \max_{x\in[-1,1]} g_{\beta,h}\left(x\right).$$

This value $m_{\beta,h}$ satisfies the mean-field equation

$$m = \tanh\left(\beta h + \beta m\right), \tag{9.3}$$

as is easily checked.

We summarize the basic results:

Theorem 9.1

a) *Let $Z_{\beta,h,N}$ be the partition function of the Curie-Weiss model as defined in (9.1). Then*

$$f\left(\beta,h\right) = \lim_{N\to\infty} \frac{1}{N} \log Z_{\beta,h,N}$$

exists is given by

$$f\left(\beta,h\right) = \sup_{x\in[-1,1]} g\left(\beta,h\right) + \log 2.$$

b) If $h \neq 0$, then $\bar{\sigma}_N$ converges in $P_{\beta,h,N}$-probability to $m_{\beta,h}$, i.e. for any $\varepsilon > 0$ one has

$$\lim_{N\to\infty} \mathcal{G}_{\beta,h,N}\left(\left|\frac{S_N}{N} - m_{\beta,h}\right| \geq \varepsilon\right) = 0.$$

c) If $h = 0$ and $\beta \leq 1$, then $\bar{\sigma}_N$ converges in $P_{\beta,0,N}$-probability to 0. If $\beta > 1$, then the $P_{\beta,0,N}$-law of S_N/N converges to

$$\frac{1}{2}\delta_{m_\beta} + \frac{1}{2}\delta_{-m_\beta}.$$

This means that for any $0 < \varepsilon < m_\beta$, one has

$$\lim_{N\to\infty} \mathcal{G}_{\beta,0,N}\left(\left|\frac{S_N}{N} - m_\beta\right| \leq \varepsilon\right) = 1/2,$$
$$\lim_{N\to\infty} \mathcal{G}_{\beta,0,N}\left(\left|\frac{S_N}{N} + m_\beta\right| \leq \varepsilon\right) = 1/2.$$

Proof. Left as an exercise. All the statements follow easily from Stirling's formula. ■

$m_{\beta,h}$ is the mean magnetization $\bar{\sigma}_N$ in the $N \to \infty$ limit under the Gibbs measure. In physics literature, the equation (9.3) is usually derived via a "cavity" argument. For that, one argues that m should be the Gibbs expectation for a single spin. By symmetry, it doesn't matter which one takes, so we take the last one:

$$m \approx E_{\beta,h,N}(\sigma_N) = \frac{\sum_\sigma \sigma_N \exp\left[-\beta\sigma_N \frac{1}{N}\sum_{j=1}^{N-1}\sigma_j - \beta h\sigma_N - \beta H_{N-1}\left(\sigma^{(N-1)}\right)\right]}{\sum_\sigma \exp\left[-\beta\sigma_N \frac{1}{N}\sum_{j=1}^{N-1}\sigma_j - \beta h\sigma_N - \beta H_{N-1}\left(\sigma^{(N-1)}\right)\right]},$$

where $\sigma^{(N-1)} = (\sigma_1, \ldots, \sigma_{N-1})$, and H_{N-1} is the Hamiltonian on the first $N-1$ spins. Summing first σ_N out in this expression, and the other ones afterwards, one gets

$$E_{\beta,h,N}(\sigma_N) = \frac{E_{\beta,h,N-1}\sinh\left(\beta\frac{1}{N}\sum_{j=1}^{N-1}\sigma_j + \beta h\right)}{E_{\beta,h,N-1}\cosh\left(\beta\frac{1}{N}\sum_{j=1}^{N-1}\sigma_j + \beta h\right)}.$$

Under the Gibbs measure on the first $N-1$ spin variables, one should have

$$\frac{1}{N}\sum_{j=1}^{N-1}\sigma_j \approx \frac{1}{N-1}\sum_{j=1}^{N-1}\sigma_j \approx m,$$

the last approximation by disregarding possible fluctuations around the mean. By this chain of arguments, one gets

$$E_{\beta,h,N}(\sigma_N) \approx \tanh(\beta h + \beta m),$$

which leads in the $N \to \infty$ limit to (9.3). In spin glass theory, there are similar equations, the TAP (for Thouless, Anderson, Palmer) equations, which however are much more delicate to discuss and prove. We come to that later on.

It is not difficult to get more information than in Theorem 9.1 out with some refinements of the arguments. For instance one can prove that in the "one-phase region", i.e. either $h \neq 0$ or $h = 0$ and $\beta \leq 1$, the spins under $\mathcal{G}_{\beta,h,N}$ behave like i.i.d. spins with possibly tilted mean. To be precise, for $m \in (-1, 1)$ consider Bernoulli measure with mean, i.e. $p_m(1) := (1+m)/2$, $p_m(-1) = 1 - p_m(1) = (1-m)/2$.

Proposition 9.2
Under the above conditions, one has for any $K \in \mathbb{N}$:

$$\lim_{N\to\infty} \mathcal{G}_{\beta,h,N}(\sigma_1 = i_1, \ldots, \sigma_K = i_K) = \prod_{j=1}^{K} p_m(i_j),$$

where $m = 0$ for $h = 0, \beta \leq 1$, and $m = m_{\beta,h}$ for $h \neq 0$.

Proof. Left as an exercise. ■

The proposition states that in the one-phase region, the Gibbs-measure under the $N \to \infty$ limit is simply coin tossing with possibly tilted mean. There is a similar statement also in the two-phase region, i.e. $\beta > 1$, $h = 0$. In that case the Gibbs distribution converges to a mixture of two tilted Bernoulli measures. Here is the statement, the proof is left is also left as an exercise:

Proposition 9.3
Assume $\beta > 1$ and $h = 0$, and let m_β be the positive solution of $m = \tanh(\beta m)$. Then for any K, and any $i_1, \ldots, i_K \in \{-1, 1\}$:

$$\lim_{N\to\infty} \mathcal{G}_{\beta,0,N}(\sigma_1 = i_1, \ldots, \sigma_K = i_K) = \frac{1}{2}\prod_{j=1}^{K} p_{m_\beta}(i_j) + \frac{1}{2}\prod_{j=1}^{K} p_{-m_\beta}(i_j).$$

These properties of the Curie-Weiss model are coming under the name "symmetry breaking". The coin tossing measures are the so-called "pure states". If $h \neq 0$ or $h = 0$ and $\beta \leq 1$, the Gibbs measure converges to a "pure state". In the case $h = 0$, $\beta > 1$, the Gibbs measure converges to a mixture of two symmetric pure states. As in that case, the relevant pure states, namely coin tossing with mean m_β and $-m_\beta$ are not symmetric under sign change, one says that the model "breaks" the symmetry, although, of course, the limiting measure is still symmetric.

Even the very simplest mean field spin glasses have a much more complicated symmetry breaking.

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