Function spaces and stochastic processes on fractals I

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Aim

Consider Besov-Lipschtiz spaces which appear as domains of Dirichlet forms on fractals.

Properties of the Besov spaces $\leftrightarrow$ Properties of the corresponding stochastic proc.

Plan

(L1) Dirichlet forms on fractals and their domains

Short survey for D-forms on fractals, characterization of their domains

(L2) Jump type processes on d-sets (Alfors d-regular sets)

Relations of various jump-type processes on d-sets, heat kernel estimates

(L3) Trace theorem for Dirichlet forms on fractals and an application

Trace theorem for self-smilar D-forms on self-smilar sets to self-similar subsets,

Application: diffusion processes penetrating fractals

1 Dirichlet forms on fractals and their domains

1.1 A quick view of the theory of Dirichlet forms

General Theory (see Fukushima-Oshima-Takeda '94 etc.)

$\{X_t\}_t$: Sym. Hunt proc. on $(K, \mu) \oplus$ cont. path (diffusion)

$\Leftrightarrow$ $-\Delta$: non-neg. def. self-adj. op. on $\mathbb{L}^2$ s.t. $P_t := \exp(t\Delta)$ Markovian $\oplus$ local

$P_t f(x) = E^x[f(X_t)], \ \lim_{t\to 0}(P_t - I)/t = \Delta$

$\Leftrightarrow$ $(\mathcal{E}, \mathcal{F})$: regular Dirichlet form (i.e. sym. closed Markovian form) on $\mathbb{L}^2$

$$\mathcal{E}(u,v) = \int_K \sqrt{-\Delta}u\sqrt{-\Delta}v d\mu, \ \mathcal{F} = \mathcal{D}(\sqrt{-\Delta}) \oplus \text{local}$$

•$(\mathcal{E}, \mathcal{F})$: regular $\overset{\text{Def}}{\Leftrightarrow}$ $\exists C \subset \mathcal{F} \cap C_0(K)$ linear space which is dense

i) in $\mathcal{F}$ w.r.t. $\mathcal{E}_1$-norm and ii) in $C_0(K)$ w.r.t. $\|\cdot\|_\infty$-norm.

•$(\mathcal{E}, \mathcal{F})$: local $\overset{\text{Def}}{\Leftrightarrow}$ $(u, v \in \mathcal{F}, \mathrm{Supp}\, u \cap \mathrm{Supp}\, v = \emptyset \Rightarrow \mathcal{E}(u,v) = 0)$.

Example

BM on $\mathbb{R}^n \Leftrightarrow$ Laplace op. on $\mathbb{R}^n \Leftrightarrow \mathcal{E}(f,f) = \frac{1}{2}\int |\nabla f|^2 dx,\ \mathcal{F} = H^1(\mathbb{R}^n)$

1.2 Sierpinski gaskets

$\{p_0, p_1, \cdots, p_n\}$: vertices of the n-dimensional simplex, p_0: the origin.

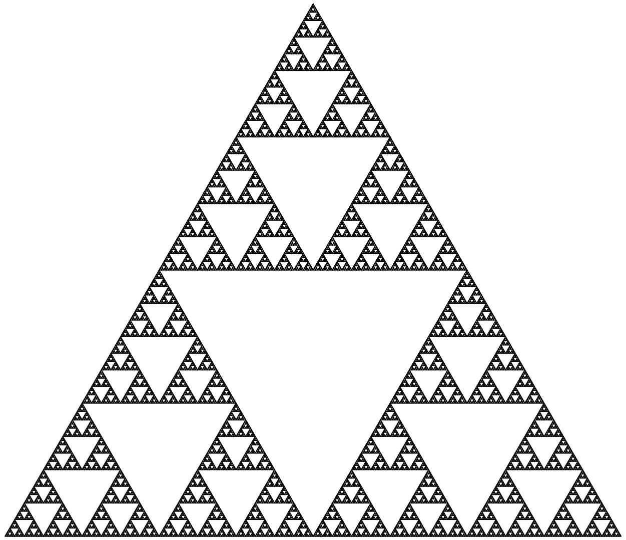
$F_i(z) = (z - p_{i-1})/2 + p_{i-1},\ z \in \mathbb{R}^n,\quad i = 1, 2, \cdots, n+1$

$\exists 1$ non-void compact set K s.t. $K = \cup_{i=1}^{n+1} F_i(K)$.

K:(n-dimensional) *Sierpinski gasket.*

When $n = 1$, $K = [p_0, p_1]$.

For simplicity, we will consider the 2-dimensional gasket.



$V_0 = \{p_0, p_1, p_2\}$, $V_n = \cup_{i_1,\cdots,i_n \in I} F_{i_1 \cdots i_n}(V_0)$

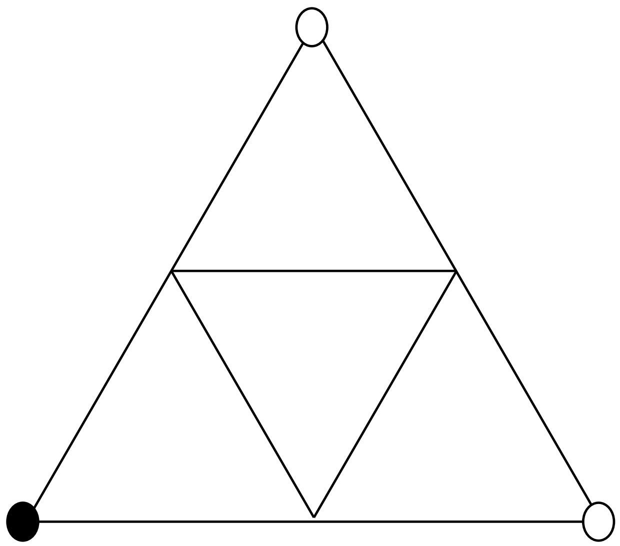
where $I := \{1, 2, 3\}$ and $F_{i_1 \cdots i_n} := F_{i_1} \circ \cdots \circ F_{i_n}$.

Let $V_* = \cup_{n \in \bar{\mathbb{N}}} V_n$, where $\bar{\mathbb{N}} := \mathbb{N} \cup \{0\}$. Then $K = Cl(V_*)$.

$d_f := \log 3 / \log 2$: Hausdorff dimension of K (w.r.t. the Euclidean metric)

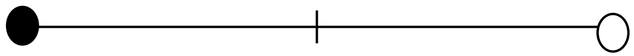
μ: (normalized) Hausdorff measure on K, i.e. a Borel measure on K s.t.

$$\mu(F_{i_1 \cdots i_n}(K)) = 3^{-n} \qquad \forall i_1, \cdots, i_n \in I.$$



$E^{\bullet}[\sigma_{\circ}] = 5$

Cf.



$E^{\bullet}[\sigma_{\circ}] = 4$

1.3 Construction of Brownian motion on the gasket (Ideas)

(Goldstein '87, Kusuoka '87) X_n: simple random walk on V_n

$$X_n([5^n t]) \xrightarrow{n\to\infty} B_t: \text{ Brownian motion on } K$$

1.4 Construction of Dirichlet forms on the gaskets

For $f, g \in \mathbb{R}^{V_n} := \{h : h \text{ is a real-valued function on } V_n\}$, define

$$\mathcal{E}_n(f,g) := \frac{b_n}{2} \sum_{i_1 \cdots i_n \in I} \sum_{x,y \in V_0} (f \circ F_{i_1 \cdots i_n}(x) - f \circ F_{i_1 \cdots i_n}(y))(g \circ F_{i_1 \cdots i_n}(x) - g \circ F_{i_1 \cdots i_n}(y)),$$

where $\{b_n\}$ is a sequence of positive numbers with $b_0 = 1$ (conductance on each bond).

Choose $\{b_n\}$ s.t. $\exists$ nice relations between the $\mathcal{E}_n$'s

Elementary computations yield

$$\inf\{\mathcal{E}_1(f,f) : f \in \mathbb{R}^{V_1},\ f|_{V_0} = u\} = \frac{3}{5} \cdot b_1 \mathcal{E}_0(u,u) \qquad \forall u \in \mathbb{R}^{V_0}. \tag{1.1}$$

So, taking $b_n = (5/3)^n$, we have

$$\mathcal{E}_n(f|_{V_n}, f|_{V_n}) \leq \mathcal{E}_{n+1}(f,f) \quad \forall f \in \mathbb{R}^{V_{n+1}}$$

("=" $\Leftrightarrow$ f is 'harmonic' on $V_{n+1} \setminus V_n$).

Define

$$\mathcal{F}_* := \{f \in \mathbb{R}^{V_*} : \lim_{n\to\infty} \mathcal{E}_n(f,f) < \infty\}, \qquad \mathcal{E}(f,g) := \lim_{n\to\infty} \mathcal{E}_n(f,g) \qquad \forall f,g \in \mathcal{F}_*.$$

$(\mathcal{E}, \mathcal{F}_*)$: quadratic form on $\mathbb{R}^{V_*}$.

Further, $\forall f \in \mathbb{R}^{V_m}$, $\exists 1 P_m f \in \mathcal{F}_*$ s.t. $\mathcal{E}(P_m f, P_m f) = \mathcal{E}_m(f,f)$.

Want: to extend this form to a form on $\mathbb{L}^2(K,\mu)$.

Define $R(p,q)^{-1} := \inf\{\mathcal{E}(f,f) : f \in V_*, f(p) = 1, f(q) = 0\} \quad \forall p,q \in V_*,\ p \neq q.$

$R(p,q)$: effective resistance between p and q. Set $R(p,p) = 0$ for $p \in V_*$.

Proposition 1.1 *1) $R(\cdot,\cdot)$ is a metric on V_*. It can be extended to a metric on K, which gives the same topology on K as the one from the Euclidean metric.*

2) For $p \neq q \in V_*$, $R(p,q) = \sup\{|f(p) - f(q)|^2/\mathcal{E}(f,f) : f \in \mathcal{F}_*, f(p) \neq f(q)\}$.

$$\text{So, } |f(p) - f(q)|^2 \leq R(p,q)\mathcal{E}(f,f), \qquad \forall f \in \mathcal{F}_*, p,q \in V_*. \tag{1.2}$$

Remark. $R(p,q) \asymp \|p-q\|^{d_w - d_f}$, where $d_w = \log 5/\log 2$ (Walk dimension).

(Here $f(x) \asymp g(x) \Leftrightarrow c_1 f(x) \le g(x) \le c_2 f(x), \quad \forall x.$)

By (1.2), $f \in \mathcal{F}_*$ can be extended conti. to K.

$\mathcal{F}$: the set of functions in $\mathcal{F}_*$ extended to $K \quad \Rightarrow \mathcal{F} \subset C(K) \subset \mathbb{L}^2(K,\mu)$.

Theorem 1.2 (Kigami) *$(\mathcal{E},\mathcal{F})$ is a local regular D-form on $\mathbb{L}^2(K,\mu)$.*

$$|f(p)-f(q)|^2 \le R(p,q)\mathcal{E}(f,f) \qquad \forall f \in \mathcal{F},\ \forall p,q \in K \tag{1.3}$$

$$\mathcal{E}(f,g) = \frac{5}{3}\sum_{i\in I}\mathcal{E}(f\circ F_i, g\circ F_i) \quad \forall f,g \in \mathcal{F} \tag{1.4}$$

$\{B_t\}$: corresponding diffusion process (Brownian motion)

Δ: corresponding self-adjoint operator on $\mathbb{L}^2(K,\mu)$.

Uniqueness (Barlow-Perkins '88) Any self-similar diffusion process on K whose law is invariant under local translations and reflections of each small triangle is a constant time change of $\{B_t\}$. — Metz, Peirone, Sabot, . . .

1.5 Properties of the Dirichlet forms on the gaskets

(A) Spectral properties (Fukushima-Shima '92) $-\Delta$ has a compact resolvent.

Set $\rho(x) = \sharp\{\lambda \leq x : \lambda \text{ is an eigenvalue of } -\Delta\}$. Then

$$0 < \liminf_{x\to\infty} \frac{\rho(x)}{x^{d_s/2}} < \limsup_{x\to\infty} \frac{\rho(x)}{x^{d_s/2}} < \infty. \tag{1.5}$$

(Barlow-Kigami '97) $<$ above is because

$\exists$ 'many' localized eigenfunctions that produce eigenvalues with high multiplicities

u: a localized eigenfunction $\overset{\text{Def}}{\Leftrightarrow}$ u: is an eigenfunction of $-\Delta$ s.t.

$$\text{Supp } u \subset O,\ \exists \text{ open set } O \subset \text{Int } K.$$

$d_s = 2\log 3/\log 5 = 2d_f/d_w$: spectral dimension

— Kigami-Lapidus, Lindstrøm, Mosco, Strichartz, Teplyaev, . . .

(B) Heat kernel estimates (Barlow-Perkins '88)

$\exists p_t(x,y)$: jointly continuous sym. transition density of $\{X_t\}$ w.r.t. μ

$(P_t f(x) = \int_K p_t(x,y) f(y)\mu(dy)\ \forall x \in K,\quad \frac{\partial}{\partial t} p_t(x_0,x) = \Delta_x p_t(x_0,x)\ )$ s.t.

$$c_1 t^{-d_s/2} \exp(-c_2(\frac{d(x,y)^{d_w}}{t})^{\frac{1}{d_w-1}}) \le p_t(x,y) \le c_3 t^{-d_s/2} \exp(-c_4(\frac{d(x,y)^{d_w}}{t})^{\frac{1}{d_w-1}}). \quad (1.6)$$

— Barlow-Bass, Hambly-K, Grigor'yan-Telcs, ...

By integrating (1.6), we have $E^0[d(0,X_t)] \asymp t^{1/d_w}$.

$d_w = \log 5/\log 2 > 2, d_s = 2\log 3/\log 5 = 2d_f/d_w < 2$

As $d_w > 2$, we say the process is sub-diffusive.

n-dim. Sierpinski gasket $(n \ge 2)$

$d_f = \log(n+1)/\log 2, d_w = \log(n+3)/\log 2 > 2, d_s = 2\log(n+1)/\log(n+3) < 2$

(C) Domains of the Dirichlet forms

For $1 \le p < \infty,\ 1 \le q \le \infty,\ \beta \ge 0$ and $m \in \bar{\mathbb{N}}$, set

$$a_m(\beta, f) := L^{m\beta}(L^{md_f} \int\int_{|x-y|<c_0 L^{-m}} |f(x) - f(y)|^p d\mu(x) d\mu(y))^{1/p}, \quad f \in \mathbb{L}^p(K, \mu),$$

where $1 < L < \infty,\ 0 < c_0 < \infty$.

$\Lambda^{\beta}_{p,q}(K)$: a set of $f \in \mathbb{L}^p(K, \mu)$ s.t. $\bar{a}(\beta, f) := \{a_m(\beta, f)\}_{m=0}^{\infty} \in l^q$.

$\Lambda^{\beta}_{p,q}(K)$ is a *Besov-Lipschitz space*. It is a Banach space.

$\underline{p = 2}$ $\Lambda^{\beta}_{2,q}(\mathbb{R}^n) = B^{\beta}_{2,q}(\mathbb{R}^n)$ if $0 < \beta < 1$, $= \{0\}$ if $\beta > 1$.

$\underline{p = 2, \beta = 1}$ $\Lambda^{1}_{2,\infty}(\mathbb{R}^n) = H^1(\mathbb{R}^n)$, $\Lambda^{1}_{2,2}(\mathbb{R}^n) = \{0\}$.

Theorem 1.3 (Jonsson '96, K, Paluba, Grigor'yan-Hu-Lau, K-Sturm)

Let $(\mathcal{E}, \mathcal{F})$ be the Dirichlet form on the gasket. Then,

$$\mathcal{F} = \Lambda^{d_w/2}_{2,\infty}(K).$$

Proof. Proof of $\mathcal{F} \subset \Lambda_{2,\infty}^{d_w/2}$. Let $\mathcal{E}_t(f,f) := (f - P_t f, f)_{\mathbb{L}^2}/t, \quad f \in \mathbb{L}^2(K,\mu)$. Then,

$$\begin{aligned}
\mathcal{E}_t(f,f) &= \frac{1}{2t}\int\int_{K\times K}(f(x)-f(y))^2 p_t(x,y)\mu(dx)\mu(dy) \\
&\geq \frac{1}{2t}\int\int_{|x-y|\leq c_0 t^{1/d_w}}(f(x)-f(y))^2 p_t(x,y)\mu(dx)\mu(dy) \\
&\geq \frac{c_1}{2t}\int\int_{|x-y|\leq c_0 t^{1/d_w}} t^{-d_s/2}(f(x)-f(y))^2\mu(dx)\mu(dy), \qquad (1.7)
\end{aligned}$$

where (1.6) was used in the last inequality.

Take $t = L^{-md_w}$ and use $d_s/2 = d_f/d_w \quad \Rightarrow \quad$ (1.7)$= c_1 a_m(d_w/2, f)^2$.

$\mathcal{E}_t(f,f) \nearrow \mathcal{E}(f,f)$ as $t \downarrow 0$. So we obtain $\sup_m a_m(d_w/2, f) \leq c_2\sqrt{\mathcal{E}(f,f)}$.

Proof of $\mathcal{F} \supset \Lambda_{2,\infty}^{d_w/2}$. Set $\gamma = 1/(d_w - 1)$, diam $(K) = 1$. Then, $\forall g \in \Lambda_{2,\infty}^{d_w/2}$

$$\begin{aligned}
\mathcal{E}_t(g,g) &= \frac{1}{2t} \int\int_{\substack{x,y\in K \\ |x-y|\le 1}} (g(x)-g(y))^2 p_t(x,y)\mu(dx)\mu(dy) \\
&\le \frac{1}{2t}\sum_{m=1}^{\infty} c_3 t^{-d_s/2} e^{-c_4(tL^{md_w})^{-\gamma}} \int\int_{L^{-m}<|x-y|\le L^{-m+1}} (g(x)-g(y))^2 \mu(dx)\mu(dy) \\
&\le c_3 t^{-(1+d_s/2)} \sum_{m=1}^{\infty} e^{-c_4(tL^{md_w})^{-\gamma}} L^{-m(d_w+d_f)} a_{m-1}(d_w/2, g)^2, \qquad (1.8)
\end{aligned}$$

where (1.6) was used in the first inequality. Let $\Phi_t(x) = e^{-c_4(tL^{xd_w})^{-\gamma}} L^{-x(d_w+d_f)}$.

- $\Phi_t(0) > 0$, $\lim_{x\to\infty} \Phi_t(x) = 0$ and $\int_0^\infty \Phi_t(x)dx = c_5 t^{1+d_s/2}$.
- $\exists x_t > 0$ s.t. $\Phi_t(x) \uparrow (0 \le \forall x < x_t)$, $\Phi_t(x) \downarrow (x_t < \forall x < \infty)$, and $\Phi_t(x_t) = c_6 t^{1+d_s/2}$.

Thus, $\sum_{m=1}^\infty \Phi_t(m) \le \int_0^\infty \Phi_t(x)dx + 2\Phi_t(x_t) \le c_7 t^{1+d_s/2}$.

Since $(1.8) \le c_3 t^{-(1+d_s/2)} (\sup_m a_m(d_w/2, f))^2 \sum_{m=1}^\infty \Phi_t(m)$,

we conclude that $\sup_{t>0} \mathcal{E}_t(g,g) = \lim_{t\to 0} \mathcal{E}_t(g,g) \le c_8 (\sup_m a_m(d_w/2, f))^2$. □

1.6 Unbounded Sierpinski gaskets

$\hat{K} := \cup_{n\geq 1} 2^n K$: the unbdd Sierpinski gasket • Construction of D-forms, as in Thm 1.2.

- Heat kernel estimates : (1.6) holds for all $x, y \in \hat{K}, 0 < t < \infty$.

- Domains of the D-forms: Thm 1.3 holds.

1.7 More general fractals

- Nested fractals (Lindstrøm '90): Similar constructions, similar results.

- P.c.f. self-similar sets (Kigami '93): Under the existence of the 'reg. harm. structure', similar constructions, generalized versions for (A), (B) and (C).

- Sierpinski carpets: Construction of D-forms, much harder, but possible (Barlow-Bass etc). Similar results for (B) and (C).