Function spaces and stochastic processes on fractals II

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2 Jump type processes on d-sets

K: compact d-set in $\mathbb{R}^n$ $(n \geq 2, 0 < d \leq n)$. I.e., $K \subset \mathbb{R}^n$, $\exists c_1, c_2 > 0$ s.t.

$$c_1 r^d \leq \mu(B(x,r)) \leq c_2 r^d \qquad \text{for all } \ x \in K,\ 0 < r \leq 1,$$

$B(x,r)$: ball center x, radius r w.r.t. Euclidean metric.

d: Hausdorff dimension of K, μ: Hausdorff measure on K.

Besov space $\mathbb{B}^{\alpha}_{2,2}(K),\ \alpha > 0$ (Triebel)

- $\mathbb{B}^{\alpha}_{2,2}(K) := tr_K H^{\alpha+(n-d)/2}(\mathbb{R}^n)$ Hilbert space
- $\|f|\mathbb{B}^{\alpha}_{2,2}(K)\| := \inf_{tr_K\, g=f} \|g|H^{\alpha+(n-d)/2}(\mathbb{R}^n)\|$, A_α: corresponding s.a. op.

Theorem 2.1 *A_α is a pos-def. s.a. op. on $L^2(K,\mu)$ with pure point spectrum.*

$$(k\text{-th eigenvalue of } A_\alpha) \asymp k^{2d/\alpha}, \qquad k \in \mathbb{N}.$$

Here, $H^\alpha(\mathbb{R}^n) = B^\alpha_{2,2}(\mathbb{R}^n)$.

For $\alpha > 0$ and $k \in \mathbb{N}$ where $k < \alpha \leq k+1$,

$$B^\alpha_{p,q}(\mathbb{R}^n) := \{u \in \mathbb{L}^p(\mathbb{R}^n, m) : \|u\|_{B^\alpha_{p,q}} < \infty\},$$

$$\text{where} \quad \|u\|_{B^\alpha_{p,q}} := \sum_{0\leq|j|\leq k} \|D^j u\|_{\mathbb{L}^p} + \sum_{|j|=k} \Big(\int_{\mathbb{R}^n} \frac{\|\Delta_h D^j u\|^q_{L^p}}{|h|^{n+q(\alpha-k)}} dh\Big)^{1/q}. \quad --(*)$$

Here, $j = (j_1, \ldots, j_n)$, $|j| = j_1 + \cdots + j_n$, $D^j = \frac{\partial^{|j|}}{\partial x_1^{j_1} \cdots \partial x_n^{j_n}}$,

and $\Delta_h f(x) := f(x+h) - f(x)$.

When $\alpha \in \mathbb{N}$, Δ_h in (*) is changed to Δ_h^2.

Three "natural" jump-type processes on d-sets

Jump process as a Besov space

For $0 < \alpha < 2$, let

$$\mathcal{E}_{Y^{(\alpha)}}(f,f) = \int\int_{K\times K} \frac{c(x,y)|u(x)-u(y)|^2}{|x-y|^{d+\alpha}}\mu(dx)\mu(dy),$$

where $c(x,y)$ is jointly measurable, $c(x,y) = c(y,x)$ and

$$c(x,y) \asymp 1.$$

A Besov space $\Lambda_{2,2}^{\alpha/2}(K)$ is defined as follows,

$$\begin{aligned} \|u|\Lambda_{2,2}^{\alpha/2}(K)\| &= \|u\|_{\mathbb{L}^2(K,\mu)} + \left(\int\int_{K\times K} \frac{|u(x)-u(y)|^2}{|x-y|^{d+\alpha}}\mu(dx)\mu(dy)\right)^{1/2} \\ \Lambda_{2,2}^{\alpha/2}(K) &= \{u : u \text{ is measurable}, \|u|\Lambda_{2,2}^{\alpha/2}(K)\| < \infty\}. \end{aligned}$$

$(\mathcal{E}_{Y^{(\alpha)}}, \Lambda_{2,2}^{\alpha/2}(K))$ is a regular Dirichlet space on $\mathbb{L}^2(K,\mu)$.

Denote $\{Y_t^{(\alpha)}\}_{t\geq 0}$ the corresponding Hunt process on K.

$\Lambda^{\alpha/2}_{2,2}(K) = \mathbb{B}^{\alpha/2}_{2,2}(K),\ 0 < \alpha < 2$, with equivalent norms.

Examples $\underline{c(x, y) \equiv 1}$ (Fukushima-Uemura '03, Stós '00)

*$K = \mathbb{R}^n \Rightarrow \{Y^{(\alpha)}_t\}$ is a α-stable process on $\mathbb{R}^n$.

*K: an open n-set $\Rightarrow \{Y^{(\alpha)}_t\}$ is a reflected α-stable process on K.

For each $f \in \mathbb{L}^1_{loc}(\mathbb{R}^n)$ and $x \in \mathbb{R}^n$, define

$$Rf(x) = \lim_{r \downarrow 0} \frac{1}{m(B(x, r))} \int_{B(x,r)} f(y)dy.$$

Proposition 2.2 (Jonsson-Wallin '84)

Let $\alpha \in \mathbb{R}$ *s.t.* $0 < \hat{\alpha} \equiv \alpha - (n - d) < 2$. *Then* $Tr_K : u \mapsto Rf$ *is a bounded linear surjection on*

$$Tr_K : H^{\alpha/2}(\mathbb{R}^n) \to \Lambda^{\hat{\alpha}/2}_{2,2}(K)$$

with a bounded linear right inverse E_K *(the extension operator).*

Jump process as a subordination of a diffusion on a fractal (Restrictive)

K: fractal with fractional diffusion $\{B_t^K\}_{t\ge 0}$, i.e.,

$$c_1 t^{-\frac{d}{d_w}} \exp(-c_2(\frac{|x-y|^{d_w}}{t}))^{\frac{1}{d_w-1}} \le p_t(x,y) \le c_3 t^{-\frac{d}{d_w}} \exp(-c_4(\frac{|x-y|^{d_w}}{t}))^{\frac{1}{d_w-1}}. \qquad (1.6)$$

(Examples: nested fractals, Sierpinski carpets)

$\{\xi_t\}_{t>0}$: strictly $(\alpha/2)$-stable subordinator $(0<\alpha<2)$.

I.e., 1-dim. non-neg. Lévy process, indep. of $\{B_t^K\}_{t\ge 0}$, $\quad E[\exp(-u\xi_t)] = \exp(-tu^{\alpha/2})$.

$\{\eta_t(u) : t>0, u\ge 0\}$: distribution density of $\{\xi_t\}_{t>0}$. Define

$$q_t(x,y) := \int_0^\infty p_u(x,y)\eta_t(u)du \qquad \text{for all } \ t>0,\ x,y\in K.$$

$\{X_t^{(\alpha)}\}_{t\ge 0}$: the subordinate process (with the transition density $q_t(x,y)$).

$$P_t^{X^{(\alpha)}} f := \mathbb{E}^{(\xi)}[P_{\xi_t}^{B^K} f] = \int_0^\infty P_s^{B^K} f \cdot \eta_t(s)ds.$$

Then, $\{X_t^{(\alpha)}\}_{t\ge 0}$ is a μ-symmetric Hunt process. (Stós '00, Bogdan-Stós-Sztonyk '02)

$(\mathcal{E}_{X^{(\alpha)}}, \mathcal{F}_{X^{(\alpha)}})$: the corresponding Dirichlet form on $\mathbb{L}^2(K, \mu)$.

Jump process as a time change (trace) of a stable process on $\mathbb{R}^n$

$\{B_t^{(\alpha)}\}_{t\geq 0}$: α-stable process $(0 < \alpha \leq 2)$ on $\mathbb{R}^n$

$(\mathcal{E}^{(\alpha)}, H^{\alpha/2}(\mathbb{R}^n))$: the corresponding Dirichlet form.

Proposition 2.3 *Let K be a d-set and μ be its Hausdorff measure. Assume $\alpha > n - d$, then μ is a smooth measure for $(\mathcal{E}^{(\alpha)}, H^{\alpha/2}(\mathbb{R}^n))$.*

(So, $\mathrm{Cap}_{\mathcal{E}^{(\alpha)}}(A) = 0 \Rightarrow \mu(A) = 0$.)

Let $\{A_t^{(\alpha)}\}_t$ be the PCAF which is in Revuz correspondence with μ.

($A_t^{(\alpha)}$ increases only when $B_t^{(\alpha)} \in K$.)

$\tau_t =: \inf\{s > 0 : A_s^{(\alpha)} > t\}, \quad Z_t^{(\alpha)} := B_{\tau_t}^{(\alpha)} \quad \Rightarrow \quad \{Z_t^{(\alpha)}\}_{t\geq 0}$: μ-sym. Hunt process.

$(\mathcal{E}_{Z^{(\alpha)}}, \mathcal{F}_{Z^{(\alpha)}})$: the corresponding Dirichlet form on $\mathbb{L}^2(K, \mu)$.

Proposition 2.4 *Assume $\alpha > n - d$. Then,*

$$P^x_{(\alpha)}(\sigma_{\mathcal{S}^{(\alpha)}} = 0) = 1 \qquad \text{for all} \quad x \in K,$$

where $\mathcal{S}^{(\alpha)}$ is a quasi-support of μ w.r.t. $\mathcal{E}_{Z^{(\alpha)}}$ and $\sigma_A := \{t > 0 : Z^{(\alpha)}_t \in A\}$.

*$\mathcal{E}_{Z^{(\alpha)}}(\cdot, \cdot) + \| \cdot \|_2^2$ coincides with the Besov norm (for $A_{(\alpha-(n-d))/2}$) given by Triebel.

*Related works: Jacob-Schilling '99, Farkas-Jacob '01, etc.

Riesz potential approach (Zähle, Hansen-Zähle)

$$I^\alpha_\mu f(x) := c_{n,\alpha+n-d} \int \frac{f(y)}{|x-y|^{d-\alpha}} d\mu(y), \quad f \in L^2, \qquad D^\alpha_\mu := (I^\alpha_\mu)^{-1},$$
$$\mathcal{E}^\alpha_\mu(f,g) := \int_K \sqrt{D^\alpha_\mu} f \sqrt{D^\alpha_\mu} g d\mu, \quad f, g \in \mathbb{B}^{2,2}_{\alpha/2}(K).$$

Then $(\mathcal{E}^\alpha_\mu, \mathbb{B}^{2,2}_{\alpha/2}(K)) = (\mathcal{E}_{Z^{(s)}}, \mathcal{F}_{Z^{(s)}})$ where $s = (\alpha + n - d)/2$.

$\sqrt{D^\alpha_\mu} = A^{1/2}_{\alpha/2}$.

Comparison of the forms (K '02, Stós '00) K: d-set, $\bar{\alpha} =: \alpha d_w/2$, $\hat{\alpha} =: \alpha - (n-d)$.

Proposition 2.5 (1) *For* $(n-d) < \alpha < 2$ *or* $\alpha = 2, n-2 < d < n$,

$$\mathcal{E}_{Z^{(\alpha)}}(f,f) \asymp \mathcal{E}_{Y^{(\hat{\alpha})}}(f,f) \qquad \text{for all } f \in \mathbb{L}^2(K,\mu).$$

(2) *Assume further that* K *has the fractional diffusion* (1.6). *Then, for* $0 < \alpha < 2$,

$$\mathcal{E}_{X^{(\alpha)}}(f,f) \asymp \mathcal{E}_{Y^{(\bar{\alpha})}}(f,f) \qquad \text{for all } f \in \mathbb{L}^2(K,\mu).$$

In particular, under the conditions,

$$\mathcal{F}_{X^{(\alpha)}} = \Lambda_{2,2}^{\bar{\alpha}/2}(K), \quad \mathcal{F}_{Z^{(\alpha)}} = \Lambda_{2,2}^{\hat{\alpha}/2}(K).$$

*Note that in general the three-type Dirichlet forms introduced are different and the corresponding processes cannot be obtained by time changes of others by PCAFs.

(Example: BM on the Sierpinski gasket)

Heat kernel estimates   Recall that for $0 < \alpha < 2$,

$$\mathcal{E}_{Y^{(\alpha)}}(f,f) := \int\int_{K\times K} \frac{c(x,y)|u(x)-u(y)|^2}{|x-y|^{d+\alpha}}\mu(dx)\mu(dy),$$

where $c(x,y)$ is jointly measurable, $c(x,y) = c(y,x)$ and $c(x,y) \asymp 1$.

Theorem 2.6 *For all* $0 < \alpha < 2$, $\exists p_t^{Y^{(\alpha)}}(x,y)$*: jointly continuous heat kernel s.t.*

$$p_t^{Y^{(\alpha)}}(x,y) \le c_1 t^{-d/\alpha} \qquad \forall t > 0.$$

Theorem 2.7 (Chen-K '03)   *For all* $0 < \alpha < 2$, $t > 0$,

$$c_1\left(t^{-d/\alpha} \wedge \frac{t}{|x-y|^{d+\alpha}}\right) \le p_t^{Y^{(\alpha)}}(x,y) \le c_2\left(t^{-d/\alpha} \wedge \frac{t}{|x-y|^{d+\alpha}}\right).$$

- Parabolic Harnack inequalities hold.
- Related works: Bass-Levin ('02)
- Thm 2.6 holds for $X^{(\alpha)}$, $Z^{(\alpha)}$ as well. Thm 2.7 holds for $X^{(\alpha)}$ as well.

• Thm 2.7 holds for processes $\hat{Y}^{(\alpha)}$ on unbounded d-sets $\hat{K}$ as well.

Corollary 2.8 (Transience, recurrence) *For* $\hat{K}$*,*

$\hat{Y}^{(\alpha)}$ *is transient iff* $d > \alpha$*,* *point recurrent iff* $d < \alpha$*.*

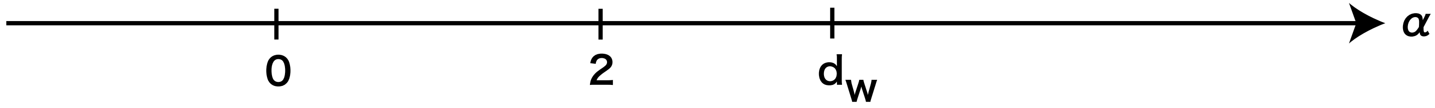
For $d = \alpha$*,* $P^x(\sigma_y < \infty) = 0, P^x(\sigma_{B(y,r)} < \infty) = 1 \;\; \forall x, y \in \hat{K}, r > 0.$

Application Hausdorff dim. for the range of the process

Proposition 2.9

$$dim_H\{\hat{Y}_t^{(\alpha)} : 0 < t < \infty\} = d \wedge \alpha \qquad \mu - a.e.$$

*More general version Y. Xiao ('04).



$$d_w := \sup\{\alpha : (\mathcal{E}_{Y(\alpha)}, \Lambda^{\alpha/2}_{2,2}(K)) \text{ is regular in } \mathbb{L}^2\}$$

Then, we can prove the former theorems for all $\alpha < d_w$

if $d_w > d$ (strongly recurrent case).

(Open Prob.) Does Them 2.6 hold $\forall \alpha < d_w$ when $d_w \le d$?

Remark: $\bar{d}_w := \sup\{\alpha : \Lambda^{\alpha/2}_{2,2}(K) \text{ is dense in } \mathbb{L}^2\}$. (cf. Paluba '00, Stós '00)

Then $d_w \le \bar{d}_w$. When there is a fractional diffusion on K, then $d_w = \bar{d}_w$.

Heat kernel estimates for jump process of mixed types (Chen-K)

$$\mathcal{E}(f,f) := \int\int_{F\times F} (u(x)-u(y))^2 J(x,y)\mu(dx)\mu(dy),$$

where $J(x,y) \geq 0$ is symmetric measurable. Assume

$$J(x,y) \asymp \frac{1}{|x-y|^d \phi(|x-y|)}. \tag{2.1}$$

Here $\phi : \mathbb{R} \to \mathbb{R}_+$ is strictly incr. and $0 < \exists\beta \leq \beta' < \infty$ s.t.

$$c_1(\frac{R}{r})^{\beta'} \leq \frac{\phi(R)}{\phi(r)} \leq c_2(\frac{R}{r})^{\beta} \qquad 0 < \forall r < R, \quad \int_{0+} \frac{r}{\phi(r)} dr < \infty. \tag{2.2}$$

Proposition 2.10

$\mathcal{D}(\mathcal{E}) := \{f \in C_0(F) : \mathcal{E}(f) < \infty\}$, $\mathcal{F} := \overline{\mathcal{D}(\mathcal{E})}^{\mathcal{E}_1}$.

$\Rightarrow$ $(\mathcal{E},\mathcal{F})$ *is a regular Dirichlet form on* $L^2(F,\mu)$.

Theorem 2.11 *Under (2.1) and (2.2),*

$\exists p_t(x,y)$*: continuous heat kernel for* $(\mathcal{E},\mathcal{F})$ *s.t.*

$$C^{-1}\left(\frac{1}{\phi^{-1}(t)^d}\wedge\frac{t}{|x-y|^d\phi(|x-y|)}\right)\le p_t(x,y)\le C\left(\frac{1}{\phi^{-1}(t)^d}\wedge\frac{t}{|x-y|^d\phi(|x-y|)}\right),$$

$0<\forall t\le c_1 \mathit{diam}\ K,\ \forall x,y\in F.$ (ϕ^{-1}*: inverse function of* ϕ)

Remark 2.12 *One can rewrite*

$$\begin{aligned}\Phi(t,|x-y|) &:= \frac{1}{\phi^{-1}(t)^d}\wedge\frac{t}{|x-y|^d\phi(|x-y|)}\\ &= \frac{1}{\phi^{-1}(t)^d}\left\{1\wedge\left(\left(\frac{\phi^{-1}(t)}{|x-y|}\right)^d\frac{t}{\phi(|x-y|)}\right)\right\}\\ &= \begin{cases}\frac{1}{\phi^{-1}(t)^d} & \text{if } \phi^{-1}(t)\ge c_*|x-y|\\ \frac{t}{|x-y|^d\phi(|x-y|)} & \text{if } \phi^{-1}(t)\le c_*|x-y|.\end{cases}\end{aligned}$$

Examples

1) $[\alpha_1, \alpha_2] \subset (0,2)$, μ: probability measure on $[\alpha_1, \alpha_2]$

$$\phi(t) := \int_{\alpha_1}^{\alpha_2} t^{\alpha}\, \nu(d\alpha).$$

Especially, $0 < \alpha_1 < \cdots < \alpha_n < 2$,

$$J(x,y) = \sum_{k=1}^{n} \frac{c_i(x,y)}{|x-y|^{d+\alpha_i}},$$

where $c^{-1} < c_i(x,y) = c_i(y,x) < c$. (cf. Elliptic Harnack: Song-Vondraček '03)

2) $[\alpha_1, \alpha_2] \subset (0,2)$, μ: probability measure on $[\alpha_1, \alpha_2]$

$$\phi(t) := \left(\int_{\alpha_1}^{\alpha_2} t^{-\alpha}\, \nu(d\alpha) \right)^{-1}.$$