Markov chain approximations to non-symmetric diffusions with bounded coefficients

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Dedicated to Professor Tadahisa Funaki on his 60th birthday.

Abstract

We consider a certain class of non-symmetric Markov chains and obtain heat kernel bounds and parabolic Harnack inequalities. Using the heat kernel estimates, we establish a sufficient condition for the family of Markov chains to converge to non-symmetric diffusions. As an application, we approximate non-symmetric diffusions in divergence form with bounded coefficients by non-symmetric Markov chains. This extends the results by Stroock-Zheng ([SZ]) to the non-symmetric divergence forms.

1 Introduction

Consider a diffusion operator in divergence form in $\mathbb{R}^d$:

$$\mathscr{L} f(x) = \frac{1}{\rho(x)} \sum_{i,j=1}^{d} \partial_{x_i}\big(a_{ij}(x)\partial_{x_j} f(x)\big), \quad f \in C^2(\mathbb{R}^d),$$

which is uniformly elliptic and bounded: the coefficients ρ, a_{ij} are real measurable functions such that

$$\text{(1.1)} \qquad \forall x \in \mathbb{R}^d, \quad \frac{1}{C_0} \le \rho(x) \le C_0, \quad |a_{ij}(x)| \le C_1,$$

and

$$\text{(1.2)} \qquad \forall x \in \mathbb{R}^d, \forall \xi \in \mathbb{R}^d, \quad \sum_{i,j=1}^{d} a_{ij}(x)\xi_i\xi_j \ge \varepsilon \sum_{i=1}^{d} \xi_i^2.$$

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Formally this corresponds a diffusion process with the non-symmetric Dirichlet form

$$\mathscr{E}(f,g) = \int_{\mathbb{R}^d} \nabla f(x) \cdot a(x) \nabla g(x) dx, \qquad f,g \in W^{1,2}(\mathbb{R}^d),$$

and the based measure $\nu(dx) = \rho(x)\,dx$. We can rewrite the operator

$$\mathscr{L} f(x) = \sum_{i,j=1}^{d} \tilde{a}_{ij}(x) \partial_{x_i} \partial_{x_j} f(x) + \sum_{i=1}^{d} b_i(x) \partial_{x_i} f(x)$$

where $\tilde{a}(x)$ is the symmetric diffusion matrix

$$\tilde{a}_{ij}(x) = \frac{1}{\rho(x)} \frac{1}{2} (a_{ij}(x) + a_{ji}(x))$$

and $b_i(x)$ is the formal drift

$$b_i(x) = \frac{1}{\rho(x)} \sum_{j=1}^{d} \partial_{x_j} a_{ij}(x).$$

Then the classical theory developed by E. De Giorgi, J. Nash and J. Moser in the fifties and sixties of the last century shows that non-negative solutions u to the calorific (parabolic) equation $(\partial_t + \mathscr{L})u = 0$ satisfy the scale-invariant parabolic Harnack principle, also the associated kernel have some Hölder regularity and with both upper and lower Gaussian bounds, cf. [SC]. The remarkable fact is that no assumption on the regularity of the coefficients is needed. This is particularly useful for example when the coefficients depend on disordered random media. These purely analytical tools can be used in order to construct the diffusion process $\{Z_t, t \geq 0\}$ on $\mathbb{R}^d$, although the standard stochastic differential equation formalism may fail, since the corresponding drift $b_i(x)$ might not even be defined.

An alternative, more probabilistic, way to construct this diffusion is the approximation by finite range random walks $\{Y_t^{(n)}, t \geq 0\}$ on the rescaled lattice $\mathscr{S}_n = \frac{1}{n}\mathbb{Z}^d$. In case of smooth coefficients $a_{ij} \in C^1(\mathbb{R}^d)$, this approach is well known cf. [SV].

The non-smooth *symmetric* case, where $a_{ij}(x) = a_{ji}(x)$ has been the object of several papers. In this situation, the diffusion process can be approximated by symmetric Markov chains. The explicit construction of [SZ] is based on a discrete analogy of the De Giorgi-Nash-Moser theory for symmetric uniformly irreducible walks which shows Hölder regularity for the corresponding rescaled heat kernels. These a priori estimates give compactness for the heat kernels and play a crucial role in the proof of the convergence. Further results allowing unbounded jump ranges converging to symmetric processes with both diffusions and jumps have been obtained by [BK] and [BKU], however all of these results so far have been restricted to the symmetric case for which the discrete theory of De Giorgi-Nash-Moser has been extensively developed.

The objective of this paper is to treat the general uniformly elliptic *non-symmetric* case under assumptions (1.1) and (1.2). We first identify a broad class of random walks on $\mathscr{S}_n$ corresponding to diffusions in divergence form with uniform elliptic coefficients as the class of random walks with *bounded cycle decomposition* sometimes also called centered walks, cf. [Birk], [Ma], see also some related work dealing with random walks on groups in [Al], [Du].

More precisely, these uniformly irreducible random walks on $\mathscr{S}_n$ admit a cycle decomposition with bounded range, bounded length of cycles and bounded jump rates (cf. Assumptions 2.1 and 2.3 below). Our first objective is to show that for such class of random walks, the diffusive scale-invariant parabolic Harnack principle holds and we derive Hölder regularity and Gaussian estimates for the corresponding heat kernel. To our knowledge, this is the first paper that shows such results in the non-symmetric setting, cf. Theorem 3.9 and Theorem 3.10 below.

This is the core of our paper. It should be noted that upper bounds of the Carne Varopoulos-type have already been obtained for the time discrete kernel by Mathieu in [Ma], from which we get heat kernel upper bounds for the time-continuous process via Poissonization procedure and Nash's inequalities.

Our derivation of the lower bounds and parabolic Harnack inequality is new. It is based on non-symmetric Dirichlet forms, the weighted Poincaré inequalities and differential inequalities, and is partially inspired by previous derivations in the symmetric case in [BK], [BKU] and [SZ]. However the lack of symmetry requires special care and new methods. A key step is played by a Jensen-type inequality (3.11) which allows us to control the non-symmetric part of the Dirichlet form.

Note that parabolic Harnack inequality also holds in the context of random walks in random environments with bounded cycle decompositions as introduced in [DK]. In view of the parabolic Harnack inequality, the quenched invariance principle proved in [DK] extends easily to a quenched local limit theorem, it suffices to apply the method presented in [BaH]; cf. Theorem 4.7.

Equipped with these regularity results for the heat kernel we can then focus on the convergence of the associated rescaled process. In particular tightness follows from the upper bound while the Hölder regularity implies the compactness of the corresponding heat kernels. We show that weak convergence of the random walks $\{Y_t^{(n)}, t \geq 0\}$ to the diffusion process $\{Z_t, t \geq 0\}$ takes place, once the coefficients of the discrete non-symmetric Dirichlet form converge locally in $L^1(\mathbb{R}^d)$ to the given matrix a_{ij}.

Although both regularity results and convergence theorem have some interest in their own, we can view them as preparation to our main result: the explicit construction of uniformly irreducible random walks with bounded cycle decomposition on $\mathscr{S}_n$ converging to the diffusion process in divergence form for given measurable uniformly elliptic bounded matrix $a : \mathbb{R}^d \ni x \mapsto (a_{ij}(x))_{1\leq i,j\leq d} \in \mathbb{R}^{d^2}$, satisfying (1.1) and (1.2).

Our concrete construction of the cycles and weights is based on a two scale methods. In particular it avoids an intermediate smoothing procedure of the matrix

$a(x)$ as proposed by [SZ] in the symmetric case. Also we give a procedure to produce a diagonalized form of symmetric matrices based on the Feshbach transformation, with no need to compute of the eigenvalues and eigenvectors. Our method is very simple and gives explicit bounds on the range and length of corresponding cycles, and thus could be easily numerically implemented, cf. Theorem 5.5 below. The paper is organized as follows. In Section 2 we give the framework with the precise definitions of bounded cycle decomposition. Section 3 deals with heat kernel estimates; The upper bound follows from Mathieu's result obtained for discrete time walks and a standard Poissonization procedure, while the lower bound is new, based on the Jensen-type key inequality in Proposition 3.7 and the weighted Poincaré inequalities. Section 4 presents the weak convergence of the random walks to the non-symmetric diffusion process in divergence form. In Section 5, for given matrix a_{ij} we construct explicitly a family of bounded cycles such that the corresponding process converges weakly to the diffusion process.

2 Framework

For $n \in \mathbb{N}$, let $\mathscr{S}_n = n^{-1}\mathbb{Z}^d$. Let $|\cdot|$ be the Euclidean norm and $B_n(x,r) := \{y \in \mathscr{S}_n : |x-y| < r\}$. Let $\mu_x^n := n^{-d}$ for all $x \in \mathscr{S}_n$ and for each $A \subset \mathscr{S}_n$, define $\mu^n(A) = \sum_{y\in A} \mu_y^n$.

We call a cycle a finite oriented sequence of points

$$\gamma = (x_0, x_1, \cdots, x_{l(\gamma)} = x_0)$$

where $x_j = (x_j^1, \cdots, x_j^d) \in \mathbb{Z}^d$ and $l(\gamma)$ is the length of the cycle. We allow cycles of the form (x_0,x_0) or (x_0,x_1,x_0). Sometimes, we identify the cycle γ with a sequence of oriented edges, namely $\gamma = ((x_0,x_1),\cdots,(x_{l(\gamma)-1},x_0))$. By writing $(x,y) \in \gamma$, we mean that the oriented edge (x,y) belongs to the cycle. We suppose that cycles are edge self-avoiding (meaning that $(x_i,x_{i+1}) = (x_j,x_{j+1})$ implies $i = j$), but we do not assume cycles are vertex self-avoiding. We define the range of the cycle γ as

$$\text{Range}\,(\gamma) := \max\{|x_i - x_{i+1}| : x_i \in \gamma\}.$$

Let $\Gamma = \{\gamma_i : i = 1,2,\cdots\}$ be a family of cycles such that $\{x \in \mathbb{Z}^d :$ there exists $\gamma \in \Gamma$ such that $x \in \gamma\} = \mathbb{Z}^d$. We define weights of cycles by a map $\alpha : \Gamma \to (0,\infty)$.

We define a quadratic form by

$$\begin{aligned}\mathscr{E}(f,g) &= \sum_{\gamma\in\Gamma} \alpha(\gamma)\mathscr{E}_\gamma(f,g)\ \forall f,g \in \mathscr{F},\\ \mathscr{F} &= \{f : \mathbb{Z}^d \to \mathbb{R} \mid \sum_{\gamma\in\Gamma} \alpha(\gamma)\mathscr{E}_\gamma(f,f) < \infty\},\end{aligned}$$

and

$$\mathscr{E}_\gamma(f,g) = \sum_{j=0}^{l(\gamma)-1} (f(x_j) - f(x_{j+1}))g(x_j).$$

Since $\sum_{j=0}^{l(\gamma)-1}(f(x_j)-f(x_{j+1}))=0$, it holds that

$$\mathscr{E}_\gamma(f,g)=\sum_{j=0}^{l(\gamma)-1}(f(x_j)-f(x_{j+1}))(g(x_j)-A) \tag{2.1}$$

for any constant A. It should be noted that two different sets of cycles and weights can give the same quadratic form $\mathscr{E}$, that is the cycle decomposition, which has been introduced for Markov chains on finite sets by G. Birkhoff in [Birk], is not unique.

For a cycle $\gamma=(x_0,x_1,\ldots,x_{l(\gamma)}=x_0)$, the reversed cycle is given by

$$\gamma^*=(x_{l(\gamma)},x_{l(\gamma)-1},\ldots,x_0=x_{l(\gamma)}).$$

Then if we set

$$\alpha(\gamma^*)=\alpha(\gamma)$$

and $\Gamma^*=\{\gamma^*:\gamma\in\Gamma\}$ we have

$$\mathscr{E}^*(f,g)=\sum_{\gamma^*\in\Gamma^*}\alpha(\gamma^*)\mathscr{E}_{\gamma^*}(f,g)=\mathscr{E}(g,f)\ \forall f,g\in\mathscr{F}.$$

Note that for each cycle γ of length at most two: $\ell(\gamma)\le 2$ we have $\gamma^*=\gamma$. In particular the form is symmetric, $\mathscr{E}(f,g)=\mathscr{E}(g,f)$ if and only if we can find a cycle decomposition with cycles of length at most two.

Also we have

$$\frac{1}{2}(\mathscr{E}(f,g)+\mathscr{E}(g,f))=\sum_{\gamma\in\Gamma}\alpha(\gamma)\tilde{\mathscr{E}}_\gamma(f,g)\ \forall f,g\in\mathscr{F},$$

where

$$\tilde{\mathscr{E}}_\gamma(f,g)=\frac{1}{2}\sum_{j=0}^{l(\gamma)-1}(f(x_j)-f(x_{j+1}))(g(x_j)-g(x_{j+1})).$$

The quadratic form on $\mathscr{S}_n$ is defined by

$$\begin{aligned}\mathscr{E}^n(f,g)&=n^{2-d}\sum_{\gamma_n\in\Gamma_n}\alpha_n(\gamma_n)\mathscr{E}_{\gamma_n}(f,g)\ \forall f,g\in\mathscr{F}^n,\\ \mathscr{F}^n&=\{f:\mathscr{S}_n\to\mathbb{R}\mid\sum_{\gamma_n\in\Gamma_n}\alpha_n(\gamma_n)\mathscr{E}_{\gamma_n}(f,f)<\infty\},\end{aligned}$$

where Γ_n is a family of (a countable number of) cycles on $\mathscr{S}_n$ and $\alpha_n(\gamma_n)\in(0,\infty)$ is the weight of the cycle γ_n. As we will see below, the factor n^{2-d} corresponds to a diffusive scaling, where space is rescaled by $\frac{1}{n}$ and time by n^2.

For each $n\in\mathbb{N}$ and $y\in\mathscr{S}_n$, let ν_y^n be a positive number and let ν^n be the measure on $\mathscr{S}_n$ defined by $\nu^n(A)=\sum_{y\in A}\nu_y^n$ for $A\subset\mathscr{S}_n$. We assume the following for the triple $(\alpha_n,\Gamma_n,\nu_n),n\in\mathbb{N}$ which we call *bounded cycle decomposition*:

Assumption 2.1. Bounded length, bounded weights and bounded range. *There exist $M_1, \cdots, M_5 < \infty$ such that the following holds for all $n \in \mathbb{N}$:*

$$\ell(\gamma_n) \leq M_1, 0 < \alpha_n(\gamma_n) \leq M_2, Range\,(\gamma_n) \leq \frac{M_3}{n} \quad \text{for all cycles } \gamma_n, \tag{2.2}$$

$$M_4 n^{-d} \leq \nu_y^n \leq M_5 n^{-d} \quad \text{for all } y \in \mathscr{S}_n. \tag{2.3}$$

In some sense, Assumption 2.1 corresponds to the boundedness condition (1.1) for the diffusion matrix.

Under Assumption 2.1, $(\mathscr{E}^n, \mathscr{F}^n)$ is a closed form that satisfies the following (strong) sector condition;

$$\mathscr{E}^n(f,g)^2 \leq C\mathscr{E}^n(f,f)\mathscr{E}^n(g,g),$$

for all $f, g \in \mathscr{F}^n$ that are compactly supported (see [Ma, Lemma 2.12]). Here C depends only on M_1 in (2.2). – It fact, $(\mathscr{E}^n, L^2(\mathscr{S}_n, \nu^n))$ is a regular (non-symmetric) Dirichlet form, but we do not use this fact explicitly (see [MR] for the general theory of non-symmetric Dirichlet forms). Let us introduce the scalar product

$$\langle f, g \rangle_{\nu^n} := \sum_{y \in \mathscr{S}_n} f(y) g(y) \nu_y^n \tag{2.4}$$

and the infinitesimal generator $\mathscr{A}^n$ such that

$$\langle \mathscr{A}^n f, g \rangle_{\nu^n} = -\mathscr{E}^n(f,g), \quad f, g \in \mathscr{F}^n.$$

That is

$$\mathscr{A}^n f(x) = \sum_{y \in \mathscr{S}_n} q^n(x,y)(f(y) - f(x))$$

with

$$q^n(x,y) = \frac{n^{2-d}}{\nu_x^n} \sum_{\gamma_n \in \Gamma_n} \alpha_n(\gamma_n) 1_{\{(x,y) \in \gamma_n\}} = \frac{n^{2-d} a_n(x,y)}{\nu_x^n}$$

and

$$a_n(x,y) := \sum_{\gamma_n \in \Gamma_n} \alpha_n(\gamma_n) 1_{\{(x,y) \in \gamma_n\}}.$$

Under our assumption, it is easy to see that $L^2(\mathscr{S}_n, \mu^n) = L^2(\mathscr{S}_n, \nu^n) \subset \mathscr{F}^n$. Also, note that from (2.2), we can deduce the following:

$$a_n(x) := \sum_{y \in \mathscr{S}_n : y \neq x} a_n(x,y) \leq M_6 \text{ for all } x \in X. \tag{2.5}$$

Indeed, since the range is uniformly bounded, each point will only have a bounded number of cycles (at most $2^{(2M_1M_3)^d}$). These cycles have at most M_1 elements and the weights are bounded by M_2, so taking $M_6 = 2^{(2M_1M_3)^d} M_1 M_2$ suffices.

Let $Y_t^{(n)}$ be the corresponding continuous time Markov chains on $\mathscr{S}_n$. In fact, $Y_t^{(n)}$ can also be constructed from a discrete time Markov chain. Let $\{X_m^{(n)}\}$ be the discrete time Markov chain defined by

$$\mathbb{P}^x(X_1^{(n)} = y) = p_1^{(n)}(x,y) = \frac{a_n(x,y)}{a_n(x)} \text{ for all } x,y \in \mathscr{S}_n. \tag{2.6}$$

Let $\{U_i^{x,n} : i \in \mathbb{N}, x \in \mathscr{S}_n\}$ be an independent sequence of exponential random variables, where the parameter for $U_i^{x,n}$ is $n^{2-d}a_n(x)/\nu_x^n$, that is independent of $\{X_m^{(n)}\}_m$, and define

$$T_0^{(n)} = 0,\ T_m^{(n)} = \sum_{k=1}^{m} U_k^{X_{k-1}^{(n)},n}. \tag{2.7}$$

Set $\widetilde{Y}_t^{(n)} = X_m^{(n)}$ if $T_m^{(n)} \le t < T_{m+1}^{(n)}$; then the laws of $\widetilde{Y}^{(n)}$ and $Y^{(n)}$ are the same, and hence $\widetilde{Y}^{(n)}$ is a realization of $Y^{(n)}$. Note that under Assumption 2.1, the mean exponential holding time at each point for $\widetilde{Y}^{(n)}$ can be controlled uniformly from above and below by n^2.

Remark 2.2. Note that under Assumption 2.1, $\{Y_t^{(n)}\}$ is conservative. Indeed, letting $\{P_m^{X^{(n)}}\}$ be the semigroup corresponding to $X^{(n)}$, $P_1^{X^{(n)}}1(x) = \sum_{y\in\mathscr{S}_n}\mathbb{P}^x(X_1^{(n)} = y) = 1$ by (2.6). So inductively we have $P_m^{X^{(n)}}1 = 1$ for all $m \in \mathbb{N}$, so that $\{X_m^{(n)}\}$ is conservative. As we mentioned above, $\{Y_t^{(n)}\}$ is a time changed process of $\{X_m^{(n)}\}$, and under Assumption 2.1, the mean exponential holding time at each point for $Y^{(n)}$ can be controlled uniformly from above and below by n^2, so we conclude $P_t^n 1 = 1$ for all $t > 0$.

We make a second important assumption, which corresponds to the uniform elliptic condition given in (1.2) for the diffusion matrix:

Assumption 2.3. Uniform Irreducibility. *There exist $\delta > 0$ and $N \ge 1$, such that for all $x \in \mathscr{S}_n$, and $i = 1,\dots,d$ we can find $k = k(x,\pm e_i) \le N$ such that*

$$p_k^{(n)}(x, x\pm e_i/n) = \mathbb{P}^x(X_k^{(n)} = x \pm e_i/n) \ge \delta,\ \forall i = 1,2,\cdots,d, \tag{2.8}$$

where e_i is a unit vector in $\mathbb{Z}^d$ whose i-th component is 1.
Moreover, there exist $M_7 > 0$ such that the following hold for all $n \in \mathbb{N}$ and $x \in \mathscr{S}_n$:

$$a_n(x) \ge M_7. \tag{2.9}$$

Remark 2.4. Given Γ_n, we can define the graph distance associated with Γ_n as follows: For each $x,y \in \mathscr{S}_n$, we write $x \sim y$ if there exists a cycle $\gamma = (x_0,\dots,x_\ell = x_0)$ such that either $x_i = x, x_{i+1} = y$ or $x_i = y, x_{i+1} = x$ holds for some $0 \le i \le l(\gamma) - 1$. That is, in view of the above if and only if $p_1^{(n)}(x,y) + p_1^{(n)}(y,x) > 0$. For each $z,w \in \mathscr{S}_n$, a path between z and w is a sequence $z = z_0, z_1, z_2, \cdots, z_m = w$ such that $z_i \sim z_{i+1}$ for $0 \le i \le m-1$. m is called the length of the path. Now for each $x,y \in \mathscr{S}_n$, let

$$d_n(x,y) = \min\{m : \omega = (\omega_0, \cdots, \omega_m) \text{ is a path between } x \text{ and } y\},$$

if $x \neq y$ and let $d_n(x,x) = 0$. This d_n is the graph distance on $\mathscr{S}_n$. Sometimes it is convenient to work with the graph distance rather than the Euclidean distance. However, since lengths of cycles are uniformly bounded, bounded range and uniform irreducible, there exist $c_1, c_2 > 0$ such that

$$c_1 d_n(x,y)/n \le |x-y| \le c_2 d_n(x,y)/n \qquad \forall x,y \in \mathscr{S}_n, \forall n \in \mathbb{N}. \tag{2.10}$$

So we will use the Euclidean distance in this paper.

Let $p^n(t,x,y)$ be the transition density for $Y_t^{(n)}$ with respect to ν^n, namely,

$$p^n(t,x,y) = \mathbb{P}^x(Y_t^{(n)} = y)/\nu_y^n.$$

Then the semigroup P_t^n

$$P_t^n(f)(x) = \sum_{y \in \mathscr{S}_n} p^n(t,x,y) f(y) \nu_y^n$$

has the infinitesimal generator $\mathscr{A}^n$:

$$\frac{d}{dt}\langle P_t^n f, g\rangle_{\nu^n} = \langle \mathscr{A}^n(P_t^n f), g\rangle_{\nu^n} = -\mathscr{E}^n(P_t^n f, g). \tag{2.11}$$

In particular the density is a solution of the backward equation: for all $y \in \mathscr{S}_n$

$$p^n(t,x,y) = \frac{1}{\nu_y^n} 1_y(x) + \int_0^t \Big(\sum_{z \in \mathscr{S}_n} q^n(x,z)(p^n(s,z,y) - p^n(s,x,y)) \Big) ds, \quad \forall x \in \mathscr{S}_n.$$

Denote $Y_t^{*,(n)}$ as the dual process of $Y_t^{(n)}$, $\mathscr{E}^{*,n}$, $\mathscr{A}^{*,n}$, $P_t^{*,n}$ be its corresponding Dirichlet form, generator and semigroup:

$$\mathscr{E}^{*,n}(f,g) = -\langle \mathscr{A}^{*,n} f, g\rangle = \mathscr{E}^n(g,f), \quad \langle P_t^n f, g\rangle_{\nu^n} = \langle f, P_t^{*,n} g\rangle_{\nu^n}. \tag{2.12}$$

That is, the Dirichlet form is expressed in terms of the reversed cycles. The corresponding heat kernel can be expressed by

$$p^{*,n}(t,x,y) = p^n(t,y,x).$$

3 Heat kernel estimates

We assume Assumptions 2.1 and 2.3 throughout this section. We derive some estimates for the transition density. We first deal with the upper bound in Section 3.1 and then prove the corresponding lower bound in Section 3.2.

3.1 Heat kernel upper bound and exit time estimates

For $p \geq 1$, define $\|f\|_{p,n}^p = \sum_{x\in\mathscr{S}_n} |f(x)|^p \mu_x^n$. For $f \in L^2(\mathscr{S}_n, \mu^n)$, let

$$\mathscr{E}_{NN}^n(f,f) = \frac{n^{2-d}}{2} \sum_{\substack{x,y\in\mathscr{S}_n \\ |x-y|=n^{-1}}} (f(x)-f(y))^2, \tag{3.1}$$

which is the Dirichlet form for the rescaled simple symmetric random walk on $\mathscr{S}_n$.

Lemma 3.1. *Under Assumptions 2.1 and 2.3, we have the following estimates.*

$$\mathscr{E}_{NN}^n(f,f) \leq c_1 \mathscr{E}^n(f,f), \quad \textit{for all } f \in \mathscr{F}^n, \tag{3.2}$$

$$p^n(t,x,y) \leq c_1 t^{-d/2}, \; p^{*,n}(t,x,y) \leq c_1 t^{-d/2}, \quad \textit{for all } x,y \in \mathscr{S}_n, t>0. \tag{3.3}$$

PROOF. First, note that under Assumption 2.3, we have

$$M_7 \leq \inf_x a_n(x) = \inf_x \sum_{y:y\neq x} a_n(x,y)$$

and therefore with $a_n(x,y) \geq M_7 p_1^{(n)}(x,y)$,

$$\mathscr{E}^n(f,f) = \frac{n^{2-d}}{2} \sum_{x,y} \tilde{a}_n(x,y)(f(x)-f(y))^2 \geq M_7 \frac{n^{2-d}}{2} \sum_{x,y} p_1^{(n)}(x,y)(f(x)-f(y))^2,$$

where $\tilde{a}_n(x,y) = (a_n(x,y)+a_n(y,x))/2$. Also, using the Cauchy-Schwarz, we have $(f(x)-f(y))^2 \leq k\sum_{i=0}^{k-1}(f(x_i)-f(x_{i+1}))^2$ where $x_0 = x, x_k = y$. So,

$$\frac{n^{2-d}}{2} \sum_{x,y} p_k^{(n)}(x,y)(f(x)-f(y))^2 \leq k^2 \frac{n^{2-d}}{2} \sum_{x,y} p_1^{(n)}(x,y)(f(x)-f(y))^2 \leq \frac{k^2}{M_7} \mathscr{E}^n(f,f).$$

Using this, in view of Assumption 2.3, we get

$$\mathscr{E}_{NN}^n(f,f) \leq \frac{1}{\delta} \sum_{k=1}^N \frac{n^{2-d}}{2} \sum_{x,y} p_k^{(n)}(x,y)(f(x)-f(y))^2 \leq c_1 \mathscr{E}^n(f,f),$$

which gives (3.2). Now using (3.2) and [BK, Proposition 3.1], there exists $c_1 > 0$ independent of n such that for any $f \in \mathscr{F}^n$,

$$\|f\|_{2,n}^{2(1+2/d)} \leq c\mathscr{E}_{NN}^n(f,f)\|f\|_{1,n}^{4/d} \leq c_1 \mathscr{E}^n(f,f)\|f\|_{1,n}^{4/d}. \tag{3.4}$$

Given (3.4), one can deduce (3.3) by a similar way as that of the case of symmetric Dirichlet forms (see, for example [CKS], for the proof of the symmetric case). □

For $r \geq n^{-1}$, let $\mathscr{E}^{n,r}$ be the Dirichlet form corresponding to $\{Y_t^{(n),r} := r^{-1} Y_{r^2 t}^{(n)}, t \geq 0\}$ with based measure $\nu^{n,r}$, where $\nu^{n,r}(A) := r^{-d}\nu^n(rA)$ for each $A \subset \mathscr{S}_{nr} := \{x/r : x \in \mathscr{S}_n\} = (nr)^{-1}\mathbb{Z}^d$. By simple computations, we have

$$\mathscr{E}^{n,r}(f,g) = (nr)^{2-d} \sum_{\gamma_n} \alpha(\gamma_n) \mathscr{E}_{r^{-1}\gamma_n}(f,g),$$

where $r^{-1}\gamma_n = (r^{-1}x_0, r^{-1}x_1 \cdots, r^{-1}x_{l(\gamma)})$. Define

$$p^{n,r}(t,x,y) := r^d p^n(r^2t, rx, ry). \tag{3.5}$$

Then $p^{n,r}(t,x,y)$ is the heat kernel for $\mathscr{E}^{n,r}$ with respect to $\nu^{n,r}$.

Lemma 3.2. *There exist $c_1, c_2 > 0$ such that for all $m, n, r \in \mathbb{N}$, the following holds.*

$$\mathbb{P}^x(\sup_{k\le m} d_n(x, X_k^{(n)}) \ge r) \le c_1 1_{\{M_3 m \ge r\}} \exp(-c_2 r^2/m).$$

PROOF. Since $X_m^{(n)}$ started at x cannot reach $z \in \mathscr{S}_n$ with $d_n(z,x) > M_3 m$, we may assume $M_3 m \ge r$. By [Ma, Theorem 2.8], we have

$$\mathbb{P}^x(X_m^{(n)} = y) \le c_1 \nu_y^n \exp(-c_2 d_n(x,y)^2/m). \tag{3.6}$$

Summing over all $y \in \mathscr{S}_n$ with $d_n(x,y) \ge r$ and noting $c_3 R^d \le \sum_{y:d_n(x,y)\le nR} \nu_y^n \le c_4 R^d$ for all $R \in \mathbb{N}$, we have

$$\mathbb{P}^x(d_n(x, X_m^{(n)}) \ge r) \le c_5 \exp(-c_6 r^2/m).$$

Now applying [BBCK, Lemma 3.8], we obtain the desired estimate. □

Proposition 3.3. *For $A > 0$ and $0 < B < 1$, there exists $t_0 = t_0(A,B) \in (0,1)$ such that for every $n \in \mathbb{N}$, $r \ge n^{-1}$ and $x \in \mathscr{S}_n$,*

$$\mathbb{P}^x\left(\sup_{t\le r^2 t_0} |Y_t^{(n)} - Y_0^{(n)}| > rA\right) = \mathbb{P}^x\left(\sup_{t\le t_0} |Y_t^{(n),r} - Y_0^{(n),r}| > A\right) \le B. \tag{3.7}$$

Further, the same estimates hold for the dual processes $Y^{*,(n)}$ and $Y^{*,(n),r}$.

PROOF. The first equality of (3.7) holds by definition of $Y_t^{(n),r}$. Let $N_t^{(n)} = \sup\{m \in \mathbb{N} : T_m^{(n)} \le t\}$ where $\{T_m^{(n)}\}_m$ is as in (2.7). Then, $Y_t^{(n)} = X_{N_t^{(n)}}^{(n)}$, and $\mathbb{E}^x[N_t^{(n)}] \le c_1 n^2 t$, $\mathbb{P}^x(N_t^{(n)} = 0) \ge \exp(-c_1 n^2 t)$ by Assumption 2.1. So for each $\lambda > 1$,

$$\begin{aligned}
\mathbb{P}^x(|Y_{r^2t}^{(n)} - Y_0^{(n)}| > rA) &= \mathbb{P}^x(Y_{r^2t}^{(n)} \notin B(x, rA)) \\
&\le \mathbb{P}^x(Y_{r^2t}^{(n)} \notin B(x,rA), N_{r^2t}^{(n)} \le \lambda r^2 t n^2) + \mathbb{P}^x(N_{r^2t}^{(n)} > \lambda r^2 t n^2) \\
&\le \mathbb{P}^x(\sup_{k\le\lambda r^2 t n^2} d_n(x, X_k^{(n)}) \ge rnA) + \frac{1}{\lambda r^2 t n^2}\mathbb{E}^x[N_{r^2t}^{(n)}] \\
&\le c_2 \exp\Big(-c_3 \frac{(rnA)^2}{[\lambda r^2 t n^2]+1}\Big) 1_{\{M_3 \lambda r^2 t n^2 \ge rnA\}} + \frac{c_1}{\lambda},
\end{aligned} \tag{3.8}$$

where we used (2.10) in the second inequality, and Lemma 3.2 in the last inequality.

Now fix A and consider first the case $r > (An)^{-1}$. In this case, since $rnA > 1$, we have

$$c_2 \exp\Big(-c_3 \frac{(rnA)^2}{[\lambda r^2 t n^2]+1}\Big) 1_{\{M_3 \lambda r^2 t n^2 \ge rnA\}} \le c_2 \exp\Big(-c_4 \frac{A^2}{\lambda t}\Big).$$

Take $\delta > 0$ small enough so that $c_2 \exp(-c_4\delta^{-1}) \le B/4$, and choose t_0 such that $t_0/(A^2\delta) < \min\{1, B/(4c_1)\}$. Then, for $t \le t_0$, choosing $\lambda = A^2\delta/t$ (which is larger than 1 by the choice of t_0), we have

$$(\text{RHS of (3.8)}) \le c_2 \exp(-c_4\delta^{-1}) + \frac{c_1 t}{A^2\delta} < B/2.$$

For the case $r \le (An)^{-1}$, since $rA \le n^{-1}$, we have

$$\begin{aligned}\mathbb{P}^x(|Y^{(n)}_{r^2t} - Y^{(n)}_0| > rA) &= \mathbb{P}^x(Y^{(n)}_{r^2t} \notin B(x, rA)) \\ &\le \mathbb{P}^x(N^{(n)}_{r^2t} > 0) = 1 - \mathbb{P}^x(N^{(n)}_{r^2t} = 0) \\ &\le 1 - \exp(-c_1 n^2 r^2 t) \le 1 - \exp(-c_1 A^{-2} t) \le B/2\end{aligned}$$

for all $t \le t_0$ by choosing t_0 such that $\exp(-c_1 A^{-2} t_0) \ge 1 - B/2$.

So for both cases, we conclude $\mathbb{P}^x(|Y^{(n)}_{r^2t} - Y^{(n)}_0| > rA) \le B/2$ for all $t \le t_0$. (Note that the choice of constants are independent of $n \in \mathbb{N}$.) Thus, applying [BBCK, Lemma 3.8], we obtain (3.7). The dual process version holds similarly by using the dual process version of Lemma 3.2. □

For $A \subset \mathscr{S}_n$ and a process Z_t on $\mathscr{S}_n$, let

$$\tau^n = \tau^n_A(Z) := \inf\{t \ge 0 : Z_t \notin A\}, \qquad T^n_A = T^n_A(Z) := \inf\{t \ge 0 : Z_t \in A\}.$$

Lemma 3.4. *Given $\delta > 0$ there exists κ such that for each $n \in \mathbb{N}$, if $x, y \in \mathscr{S}_n$ and $C \subset \mathscr{S}_n$ with $\operatorname{dist}(x, C)$ and $\operatorname{dist}(y, C)$ both larger than $\kappa t^{1/2}$ where $t \ge n^{-1}$, then*

$$\mathbb{P}^x(Y^{(n)}_t = y, T^n_C \le t) \le \delta t^{-d/2} n^{-d}.$$

PROOF. The proof is similar to that of [BK, Lemma 4.5], but some minor changes are needed due to the non-symmetry of the process. We sketch the proof, mentioning where to modify because of the lack of symmetry.

First, using (3.3), (3.7) and the strong Markov property, we have

$$\mathbb{P}^x(Y^{(n)}_t = y, T^n_C \le t/2) \le \frac{\delta}{2}(tn^2)^{-d/2}. \tag{3.9}$$

We next consider $\mathbb{P}^x(Y^{(n)}_t = y, t/2 \le T^n_C \le t)$. If $S_C = \sup\{s \le t : Y^{(n)}_s \in C\}$, then clearly

$$\mathbb{P}^x(Y^{(n)}_t = y, t/2 \le T^n_C \le t) \le \mathbb{P}^x(Y^{(n)}_t = y, t/2 \le S_C \le t).$$

Using the dual of the heat kernel p, the following equality holds (cf. [BK, (4.7)] for symmetric case).

$$\mathbb{P}^x(Y^{(n)}_t = y, t/2 \le S_C \le t) = \mathbb{P}^y(Y^{*,(n)}_t = x, T^n_C(Y^{*,(n)}) \le t/2)\frac{\nu^n_y}{\nu^n_x}. \tag{3.10}$$

Now, arguing similarly to the proof of (3.9) and using (2.3), the right hand side of (3.10) is bounded from above by $\frac{\delta}{2}(tn^2)^{-d/2}$, and combining with (3.9) proves the proposition. □

3.2 Lower bounds and regularity for the heat kernel

In order to establish the lower bound, we use a weighted Poincaré inequality and a differential inequality along the lines of [SZ, BK, BKU]. Since our process is non-symmetric, we need a new inequality (Proposition 3.7 ii)) to establish the differential inequality. We also use (3.3), Proposition 3.3 and the dual process.

The next proposition provides lower bounds for the heat kernel killed on exiting balls and is the key step for the proof of the Hölder continuity of $p^n(t,x,y)$.

Proposition 3.5. *There exist $c_1 > 0$ and $\theta \in (0,1)$ such that for each $n \in \mathbb{N}$, if $|x-x_0|, |y-x_0| \le t^{1/2}$, $x,y,x_0 \in \mathscr{S}_n$, $t \ge n^{-1}$ and $r \ge t^{1/2}/\theta$, then*

$$\mathbb{P}^x(Y_t^{(n)} = y, \tau^n_{B(x_0,r)} > t) \ge c_1 t^{-d/2} n^{-d}.$$

To prove this we first need some preliminary lemmas. Noting (2.3), the proof of the following weighted Poincaré inequality is almost the same as in [SZ, Lemma 1.19] and [BK, Lemma 4.3].

Lemma 3.6. *Let*

$$g_n(x) = c_1(n) \prod_{i=1}^d e^{-|x_i|} \qquad x \in \mathscr{S}_n,$$

where $c_1(n)$ is determined by the equation $\sum_{x \in \mathscr{S}_n} g_n(x) \nu^n_x = 1$. Then there exists $c_2 > 0$ such that

$$c_2 \Big\langle (f - \langle f \rangle_{g_n,\nu^n})^2 \Big\rangle_{g_n,\nu^n} \le n^{2-d} \sum_{l \in \mathscr{S}_n} g_n(l) \sum_{i=1}^d \Big(f(l+\frac{e_i}{n}) - f(l) \Big)^2, \qquad f \in L^2(\mathscr{S}_n),$$

where

$$\langle f \rangle_{g_n,\nu^n} = \sum_{l \in \mathscr{S}_n} f(l) g_n(l) \nu^n_l.$$

Since our process is non-symmetric, we cannot apply the usual proof of near diagonal heat kernel lower bounds for symmetric processes directly. The next lemma plays the key role to overcome the difficulty of non-symmetry in proving a function inequality (3.20) in Proposition 3.8.

Proposition 3.7. *i) For each $l \in \mathbb{N}$, there exist $c_1 > 0$ such that*

$$\frac{1}{l} \sum_{j=1}^l e^{\alpha_j} - \frac{c_1}{l} \sum_{j=1}^l \alpha_j^2 \ge 1, \tag{3.11}$$

for all $(\alpha_1, \cdots, \alpha_l) \in \mathbb{R}^l$ with $\sum_j \alpha_j = 0$.
ii) For each $l \in \mathbb{N}$ and $M > 0$, there exist $c_2, c_3 > 0$ such that

$$\sum_{j=1}^l (e^{\alpha_j} - 1) e^{\bar{w}_j/n} + c_2/n^2 - c_3 \sum_{j=1}^l \alpha_j^2 \ge 0, \tag{3.12}$$

for all $n \ge 1$, $(\alpha_1, \cdots, \alpha_l), (\bar{w}_1, \cdots, \bar{w}_l) \in \mathbb{R}^l$ with $\sum_j \alpha_j = \sum_j \bar{w}_j = 0$ and $\max_j |\bar{w}_j| \le M$.

PROOF. i) We will prove (3.11) for $l+1$ instead of l. Since $\alpha_{l+1} = -\sum_{j=1}^{l}\alpha_j$, we need to prove

$$F(\alpha_1,\cdots,\alpha_l) := \sum_{j=1}^{l} e^{\alpha_j} + e^{-\sum_{j=1}^{l}\alpha_j} - c_1\sum_{j=1}^{l}\alpha_j^2 - c_1(\sum_{j=1}^{l}\alpha_j)^2 \geq l+1$$

for all $(\alpha_1,\cdots,\alpha_l)\in\mathbb{R}^l$. It is easy to see that if one of $\alpha_1,\cdots,\alpha_l,(-\sum_{j=1}^{l}\alpha_j)$ goes to ∞, then $F(\alpha_1,\cdots,\alpha_l)$ goes to ∞ and $F(\alpha_1,\cdots,\alpha_l)$ is continuous. So the infimum of $F(\alpha_1,\cdots,\alpha_l)$ over all $\mathbb{R}^l$ is attained at least at one point. Let $(a_1,\cdots,a_l)$ be one of such points. Then,

(3.13)
$$\partial_{x_i}F(a_1,\cdots,a_l) = g(a_i) - g(-\sum_j a_j) = 0,\ \forall i\in\{1,2,\cdots,l\},\ \text{where } g(x) = e^x - 2c_1x,$$

so $g(a_i) = g(-\sum_j a_j)$ for all i. Note that $g(x)$ attains the global minimum at $x = \log(2c_1)$, and if $g(x) = g(y)$ for $x\leq y$, we have either $x = y$ or $x < \log(2c_1) < y$. Next,

$$\partial_{x_i}\partial_{x_k}F(a_1,\cdots,a_l) = \delta_{ik}(e^{a_i} - 2c_1) + e^{-\sum_j a_j} - 2c_1,\ \forall i,k\in\{1,2,\cdots,l\},$$

so defining $H_F(a)$ as a Hessian matrix of F at $a = (a_1,\cdots,a_l)$, we have

(3.14)
$$v^tH_F(a)v = \sum_j(e^{a_j} - 2c_1)v_j^2 + (e^{-\sum_j a_j} - 2c_1)(\sum_j v_j)^2 \geq 0,\ \forall v = (v_1,\cdots,v_l)\in\mathbb{R}^l.$$

Taking $v_i = -v_j$ and $v_k = 0$ for $k\neq i,j$ in (3.14), we have

$$e^{a_i} - 2c_1 + e^{a_j} - 2c_1 \geq 0. \tag{3.15}$$

Also, taking $v_i = 1$ and $v_k = 0$ for $k\neq i$ in (3.14), we have

$$e^{a_i} - 2c_1 + e^{-\sum_j a_j} - 2c_1 \geq 0. \tag{3.16}$$

Now suppose there exists i such that $a_i \leq \log(2c_1)$. Without loss of generality, we may assume $i = 1$. Then, by (3.15) and (3.16), we have $a_k \geq \log(2c_1)$ for $k\neq 1$ and $-\sum_j a_j \geq \log(2c_1)$. By (3.13) and the property of g mentioned above, we have $a_2 = \cdots = a_l = -\sum_j a_j =: s$. Now define $T:\mathbb{R}^l\to\mathbb{R}^l$ by $T(x) = (-\sum_j x_j, x_2,\cdots,x_l)$. Then clearly $T\circ T$ is an identity map and $F(x) = F(T(x))$. So, $T(a) = (s,\cdots,s)$ also attains the minimum of F and $s\geq\log(2c_1)$. We can obtain the same conclusion if $a_i\geq\log(2c_1)$ for all i. Therefore, we conclude that there is a point $(s,\cdots,s)\in\mathbb{R}^l$ with $s\geq\log(2c_1)$ that attains the minimum of F.

Define $f(s) := F(s,\cdots,s) = le^s - e^{-ls} - c_1ls^2 - c_1l^2s^2$ for $s\geq\log(2c_1)$. By easy calculations, we see that $f'''(s) = 0$ when $s = 2(l+1)^{-1}\log l$ and $f''(x)\geq f''(2(l+1)^{-1}\log l) = (l+1)l^{2/(l+1)} - 2c_1l(l+1)$. So by choosing $c_1\leq 2^{-1}l^{-(l-1)/(l+1)}$, $f''(x)\geq 0$ for all $x\geq\log(2c_1)$. This means $f'(x)$ is monotone increasing. As $f'(0) = 0$, $f'(z)\leq 0$ for $\log(2c_1)\leq z\leq 0$ and $f'(y)\geq 0$ for $y\geq 0$. Thus, $f(s)\geq f(0) = l+1$ so we obtain the desired result.

ii) Denote the left hand side of (3.12) as $\psi(\alpha)$ where $\alpha = (\alpha_1, \cdots, \alpha_l)$, and let $\tilde{\alpha}_i = \alpha_i + \bar{w}_i/n$. Then we have

$$\begin{aligned}\psi(\alpha) &= \sum_j e^{\tilde{\alpha}_j} - l - (\sum_j e^{\frac{\bar{w}_j}{n}} - l) + \frac{c_2}{n^2} - 2c_3 \sum_j \tilde{\alpha}_j^2 + c_3(\sum_j \tilde{\alpha}_j^2 + \frac{2}{n}\sum_j \bar{w}_j\tilde{\alpha}_j - \frac{1}{n^2}\sum_j \bar{w}_j^2) \\ &\geq -(\sum_j e^{\frac{\bar{w}_j}{n}} - l) + \frac{c_2}{n^2} + c_3\sum_j(\tilde{\alpha}_j + \frac{\bar{w}_j}{n})^2 - 2c_3\sum_j \frac{\bar{w}_j^2}{n^2},\end{aligned}$$

where i) is used with $c_3 = c_1/2$. Now, since $\sum_j \bar{w}_j = 0$, there exists $c_M > 0$ such that $\sum_j e^{\frac{\bar{w}_j}{n}} - l \leq c_M \sum_j \bar{w}_j^2/n^2$. So

$$\psi(\alpha) \geq -c_M \sum_j \frac{\bar{w}_j^2}{n^2} + \frac{c_2}{n^2} - 2c_3 \sum_j \frac{\bar{w}_j^2}{n^2} \geq \frac{1}{n^2}\{c_2 - (c_M + 2c_3)lM^2\} \geq 0,$$

by taking $c_2 \geq (c_M + 2c_3)lM^2$, and the proof is completed. □

Given the above lemma, the following key estimate can be proved by some modifications of the proof of [BK, Lemma 4.4].

Proposition 3.8. *There is an* $\varepsilon > 0$ *such that*

$$p^n(t,x,y) \geq \varepsilon t^{-d/2}, \tag{3.17}$$

for all $n \in \mathbb{N}$, $(t,x,y) \in (n^{-1},\infty) \times \mathscr{S}_n \times \mathscr{S}_n$ *with* $|x-y| \leq 2t^{1/2}$.

PROOF. It is enough to prove the following: there is an $\varepsilon > 0$ such that

$$\sum_{l\in\mathscr{S}_{nr}} \log\left(p^{n,r}(\tfrac{1}{2},k,l+m)\right) g_{nr}(l)\nu_l^{n,r} \geq \tfrac{1}{2}\log\varepsilon, \tag{3.18}$$

$$\sum_{l\in\mathscr{S}_{nr}} \log\left(p^{*,n,r}(\tfrac{1}{2},k,l+m)\right) g_{nr}(l)\nu_l^{n,r} \geq \tfrac{1}{2}\log\varepsilon, \tag{3.19}$$

for any $n \in \mathbb{N}$, $r \geq n^{-1}$ and $k,m \in \mathscr{S}_n$ with $|k-m| \leq 2$. Indeed, by the Chapman-Kolmogorov equation and the fact that $g_{nr}(j) \leq 1$ for all $k,m \in \mathscr{S}_{nr}$,

$$p^{n,r}(1,k,m) \geq \sum_{j\in\mathscr{S}_{nr}} p^{n,r}(\tfrac{1}{2},k,j+k)p^{*,n,r}(\tfrac{1}{2},m,j+k)g_{nr}(j)\nu_j^{n,r}.$$

Taking logarithm on both side, by Jensen's inequality,

$$\log p^{n,r}(1,k,m) \geq \sum_{j\in\mathscr{S}_{nr}} \log\left(p^{n,r}(\tfrac{1}{2},k,j+k)\right)\nu_j^{n,r} + \sum_{j\in\mathscr{S}_{nr}} \log\left(p^{*,n,r}(\tfrac{1}{2},m,j+k)g_{nr}(j)\right)\nu_j^{n,r}.$$

Thus, (3.18) and (3.19) yield

$$r^d p^n(r^2,rk,rl) = p^{n,r}(1,k,l) \geq \varepsilon \qquad n \geq 1, |k-l| \leq 2.$$

Taking $t = r^2$, this gives (3.17).

So it is enough to prove (3.18) and (3.19). Since the arguments are parallel, we only prove (3.18). Let $k,m \in \mathscr{S}_n$ satisfy $|k-m| \le 2$ and set $u_t(l) = p^{n,r}(t,k,l+m)$. Define

$$G(t) = \sum_{l\in\mathscr{S}_{nr}} \log(u_t(l)) g_{nr}(l) \nu_l^{n,r}.$$

By Jensen's inequality, we see that $G(t) \le 0$. Further, by (2.11) and (2.12),

$$G'(t) = \sum_{l\in\mathscr{S}_{nr}} \frac{\partial u}{\partial t}(l)\frac{g_{nr}(l)}{u_t(l)}\nu_l^{n,r} = -\mathscr{E}^{*,n,r}(u_t,\frac{g_{nr}}{u_t}) = -(nr)^{2-d}\sum_{\gamma_n^*}\alpha(\gamma_n^*)\mathscr{E}_{r^{-1}\gamma_n^*}(u_t,\frac{g_{nr}}{u_t}).$$

Write $r^{-1}\gamma_n^* = (x_0,x_1,\cdots,x_{l(\gamma_n)} = x_0)$ and define

$$F_{\gamma_n} = \frac{1}{l(\gamma_n)}\sum_{j=1}^{l(\gamma_n)}\sum_{k=1}^{d}|x_j^k|,\ \bar{w}_j = nr(F_{\gamma_n} - \sum_{k=1}^{d}|x_j^k|),$$

where x_j^k is the k-th coordinate of x_j. Note that there exists M which is independent of γ_n^* such that $\sup_j|\bar{w}_j| \le M$ due to (2.2), and $\sum_{j=0}^{l(\gamma_n)-1}\bar{w}_j = 0$. Further, $g_{nr}(x_j) = c_1(rn)e^{-F_{\gamma_n}}e^{\bar{w}_j/(nr)}$. Applying Proposition 3.7 ii) with $\alpha_i = \log u_t(x_{i+1}) - \log u_t(x_i)$, we have

$$\begin{aligned}
-\mathscr{E}_{r^{-1}\gamma_n^*}(u_t,\frac{g_{nr}}{u_t}) &= \sum_{i=0}^{l(\gamma_n)-1}\frac{u_t(x_{i+1})}{u_t(x_i)}g_{nr}(x_i) - \sum_{i=0}^{l(\gamma_n)-1}g_{nr}(x_i)\\
&= c_1(rn)e^{-F_{\gamma_n}}\sum_{i=0}^{l(\gamma_n)-1}(\frac{u_t(x_{i+1})}{u_t(x_i)} - 1)e^{\bar{w}_i/(nr)}\\
&\ge c_1'e^{-F_{\gamma_n}}\Big(c_3\sum_i(\log u_t(x_{i+1}) - \log u_t(x_i))^2 - c_2/(nr)^2\Big)\\
&\ge c_1'e^{-F_{\gamma_n}}\Big(c_4\sum_i(\log u_t(x_{i+1}) - \log u_t(x_i))^2 e^{\bar{w}_i/(nr)} - c_2/(nr)^2\Big),
\end{aligned}$$

where $c_1(nr) \ge c_1'$ for all nr is used in the first inequality, and $nr \ge 1$ and $\sup_j|\bar{w}_j| \le M$ are used in the last inequality. Thus, since $\alpha(\gamma_n^*) \le M$ for any cycle γ_n^* (due to

Assumption 2.1 i)), we have

$$
\begin{aligned}
G'(t) &= -(nr)^{2-d}\sum_{\gamma_n^*}\alpha(\gamma_n^*)\mathscr{E}_{r^{-1}\gamma_n^*}(u_t,\frac{g_{nr}}{u_t})\\
&\ge (nr)^{2-d}\sum_{\gamma_n^*}\alpha(\gamma_n^*)c_1'e^{-F_{\gamma_n}}\Big(c_4\sum_i(\log u_t(x_{i+1})-\log u_t(x_i))^2e^{\bar{w}_i/(nr)}-c_2/(nr)^2\Big)\\
&\ge c_5(nr)^{2-d}\sum_{\gamma_n^*}\alpha(\gamma_n^*)\sum_i(\log u_t(x_{i+1})-\log u_t(x_i))^2 g_{nr}(x_i)-c_6M\sum_{\gamma_n^*}e^{-F_{\gamma_n}}/(nr)^d\\
&\ge c_7(nr)^{2-d}\sum_{\gamma_n^*}\alpha(\gamma_n^*)\sum_i(\log u_t(x_{i+1})-\log u_t(x_i))^2 g_{nr}(x_i)-c_8\\
&\ge c_9(nr)^{2-d}\sum_{l\in\mathscr{S}_{nr}}\sum_{j=1}^{d}\Big(\log u_t\Big(l+\frac{e^j}{nr}\Big)-\log u_t(l)\Big)^2 g_{nr}(l)-c_8
\end{aligned}
$$

(3.20)

$$
\ge c_{10}(nr)^{-d}\sum_{l\in\mathscr{S}_{nr}}(\log u_t(l)-G(t))^2 g_{nr}(l)-c_8.
$$

Here the fourth inequality can be verified similarly to the proof of (3.2), and we used Lemma 3.6 in the last inequality.

Given these estimates and (3.3), (3.7), the rest of the proof is very similar to that of [BK, Lemma 4.4], so we omit it. □

PROOF OF PROPOSITION 3.5. We have from Proposition 3.8 and (2.3) that there exists ε such that

$$
\mathbb{P}^x(Y_t^{(n)}=y)=p^n(t,x,y)\nu_y^n\ge\varepsilon t^{-d/2}n^{-d} \tag{3.21}
$$

if $|x-y|\le 2t^{1/2}$. If we take $\delta=\varepsilon/2$ and $C=B_n(x_0,r)^c$ in Lemma 3.4, then provided $r>(\kappa+1)t^{1/2}$, we have

$$
\mathbb{P}^x(Y_t^{(n)}=y,\tau^n_{B_n(x_0,r)}\le t)\le\frac{\varepsilon}{2}t^{-d/2}n^{-d}. \tag{3.22}
$$

Subtracting (3.22) from (3.21), we have

$$
\mathbb{P}^x(Y_t^{(n)}=y,\tau^n_{B_n(x_0,r)}> t)\ge\frac{\varepsilon}{2}t^{-d/2}n^{-d}
$$

if $|x-y|\le t^{1/2}$, which is equivalent to what we want. □

We introduce the space-time process $Z_s^{(n)}:=(U_s,Y_s^{(n)})$, where $U_s=U_0+s$. The filtration generated by $Z^{(n)}$ satisfying the usual conditions will be denoted by $\{\widetilde{\mathscr{F}_s};s\ge 0\}$. The law of the space-time process $s\mapsto Z_s^{(n)}$ starting from (t,x) will be denoted by $\mathbb{P}^{(t,x)}$. We say that a non-negative Borel measurable function $q(t,x)$ on $[0,\infty)\times\mathscr{S}_n$ is *parabolic* in a relatively open subset B of $[0,\infty)\times\mathscr{S}_n$ if for every relatively compact open subset B_1 of B, $q(t,x)=\mathbb{E}^{(t,x)}\Big[q(Z^{(n)}_{\tau^n_{B_1}})\Big]$ for every $(t,x)\in B_1$, where $\tau^n_{B_1}=\inf\{s>0:Z_s^{(n)}\notin B_1\}$.

We denote $T_0 := t_0(1/2,1/2) < 1$ the constant in (3.7) corresponding to $A = B = 1/2$. For $(t,x) \in [0,\infty) \times \mathscr{S}_n$ and $r > 0$, we define

$$Q^n(t,x,r) := [t, t+T_0 r^2] \times B_n(x,r),$$

where $B_n(x,r) = \{y \in \mathscr{S}_n : |x-y| < r\}$.

Given the estimate in Proposition 3.5, one can prove the uniform Hölder continuity of the heat kernel $p^n(t,x,y)$ by standard arguments.

Theorem 3.9. *There are constants $c > 0$ and $\beta > 0$ (independent of R,n) such that for every $0 < R < \infty$, every $n \geq 1$, and every bounded parabolic function q in $Q^n(0,x_0,4R)$,*

$$|q(s,x) - q(t,y)| \leq c\,\|q\|_{\infty,R} R^{-\beta} \left(|t-s|^{1/2} + |x-y|\right)^{\beta} \tag{3.23}$$

holds for (s,x), $(t,y) \in Q^n(0,x_0,R)$, where $\|q\|_{\infty,R} := \sup_{(t,y)\in[0,(4R)^2]\times\mathscr{S}_n} |q(t,y)|$. In particular, for the transition density function $p^n(t,x,y)$ of $Y^{(n)}$,
(3.24)

$$|p^n(s,x_1,y_1) - p^n(t,x_2,y_2)| \leq c\, t_0^{-(d+\beta)/2} \left(|t-s|^{1/2} + |x_1-x_2| + |y_1-y_2|\right)^{\beta},$$

for any $n^{-1} < t_0 < 1$, t, $s \geq t_0$ and $(x_i,y_i) \in \mathscr{S}_n \times \mathscr{S}_n$ with $i = 1,2$.

PROOF. Given Proposition 3.5, there are at least two ways to prove this. One way of the proof is to show the oscillation inequality and then use it iteratively to prove the uniform Hölder continuity. See for example [FS] Section 3 or [BGK]. (Note that the symmetry of the process is not used in the proof.) The other way of the proof is to follow the arguments in [BK]. Corollary 4.6 and Lemma 4.7 in [BK] can be proved exactly in the same way. Using them, Theorem 3.9 can be proved similarly to the proof of [BK, Theorem 4.9]. (In fact, the proof is easier since [BK] handles Markov chains with unbounded range of jumps whereas here jumps are all bounded.) □

Given Proposition 3.5, one can also prove the uniform Gaussian heat kernel estimates and the uniform parabolic Harnack inequality.

Theorem 3.10. *1) There exist fixed constants $C_1, \cdots, C_4 > 0$ such that*

$$C_1 t^{-d/2} \exp\Big(-C_2 \frac{|x-y|^2}{t}\Big) \leq p^n(t,x,y) \leq C_3 t^{-d/2} \exp\Big(-C_4 \frac{|x-y|^2}{t}\Big), \tag{3.25}$$

for all $n \in \mathbb{N}$, $(t,x,y) \in [n^{-1},\infty) \times \mathscr{S}_n \times \mathscr{S}_n$ with $|x-y| \leq tn$.

2) There exist fixed constants $C_1, C_2 > 0$ such that the following holds for all $n \in \mathbb{N}$. If $u = u(t,x)$ is a non-negative parabolic function on $Q^n(0,x_0,R)$, then

$$\sup_{(t,x)\in Q^n(T_0R^2/4,x_0,R/2)} u(t,x) \leq C_2 \inf_{(t,x)\in Q^n(3T_0R^2/4,x_0,R/2)} u(t,x), \tag{3.26}$$

for all $R \in [n^{-1},\infty)$ and all $x \in \mathscr{S}_n$.

PROOF. 1) The upper bound is a consequence of (3.3) and [Ma, Theorem 2.10]. It also follows from (3.3), (3.6) and the following relation which is due to (2.3):

$$p^n(t,x,y) \le e^{-c_1 n^2 t} \sum_{m=1}^{\infty} \frac{(c_2 n^2 t)^m}{m!} \mathbb{P}^x(X_m^{(n)} = y)/\nu_y^n. \tag{3.27}$$

The lower bound follows from (3.17) and the usual chain argument (see for example, [FS, Theorem 2.7] or [BGK] – similar arguments work for non-symmetric case as well).

2) Given Proposition 3.5, one can prove the parabolic Harnack inequality similarly to [FS, Section 3] or [BGK]. In fact, the equivalence of (3.25) and (3.26) are well-known in a general context (see for example, [BGK] – similar arguments work for non-symmetric case as well). □

4 Weak convergence of the process

In view of both heat kernel estimates and regularity, it is clear that tightness holds for the law of the processes. In order to identify the limiting process, we need more detailed investigations. We adapt here the method introduced in [BK, BKU] to the non-symmetric situation.

For a cycle $\gamma = (x_1, x_2, \cdots, x_{l+1} = x_0)$, let $l(\gamma) = l$ and $z_+ := x_{i+1}$ when $z = x_i$. For each $f, g \in L^2(\mathscr{S}_n, \nu^n)$,

$$\begin{aligned}
\mathscr{E}^n(f,g) &= n^{2-d} \sum_{\gamma \in \Gamma_n} \alpha(\gamma) \sum_{j=0}^{l(\gamma)-1} (f(x_j) - f(x_{j+1}))g(x_j) \\
&= n^{2-d} \sum_{\gamma \in \Gamma_n} \alpha(\gamma) \sum_{j=0}^{l(\gamma)-1} (f(x_j) - f(x_{j+1}))(g(x_j) - \frac{1}{l(\gamma)} \sum_{k=0}^{l(\gamma)-1} g(x_k)) \\
&= n^{2-d} \sum_{\gamma \in \Gamma_n} \frac{\alpha(\gamma)}{l(\gamma)} \sum_{j=0}^{l(\gamma)-1} \sum_{k=0}^{l(\gamma)-1} (f(x_j) - f(x_{j+1}))(g(x_j) - g(x_k)) \\
&= n^{2-d} \sum_{x \in \mathscr{S}_n} \sum_{\substack{\gamma \in \Gamma_n \\ \gamma \ni x}} \frac{\alpha(\gamma)}{l(\gamma)} \sum_{\substack{y \in \mathscr{S}_n \\ y \in \gamma}} (f(x) - f(x_+))(g(x) - g(y)).
\end{aligned}$$

Starting from this form, we will show the weak convergence under Assumption 4.3 below.

First, if g is defined on $\mathbb{R}^d$, we define $R_n(g)$ to be the restriction of g to $\mathscr{S}_n$:

$$R_n(g)(x) = g(x), \qquad x \in \mathscr{S}_n.$$

If g is defined on $\mathscr{S}_n$, we define $E_n g$ to be the extension of g to $\mathbb{R}^d$ defined by

$$E_n g(x) = g([x]_n),$$

where $[x]_n = ([nx_1]/n, [nx_2]/n, \ldots, [nx_d]/n)$ for $x = (x_1, x_2, \ldots, x_d) \in \mathbb{R}^d$.

We now specify some notation in order to make a precise statement of our convergence theorem. For $n \in \mathbb{N}$, set

$$|x-y|_n := n|x_1-y_1|+n|x_2-y_2|+\cdots+n|x_d-y_d| \quad \text{for } x,y \in \mathscr{S}_n.$$

Note that $1 \le |x-y|_n \le dn|x-y|$ holds for any $x,y \in \mathscr{S}_n$ with $x \ne y$, where $|x-y|$ is the Euclidean distance between x and y. Clearly $|x-y|_n$ is always a non-negative integer.

Let $\alpha_i = e_i$ if $i = 1,2,\dots,d$ and $\alpha_i = -e_{i-d}$ if $i = d+1,\dots,2d$. *A nearest neighbor path σ from x to y* is a sequence of points $p_i \in \mathscr{S}_n$ for $i = 0,1,2,\dots,k$ ($k \ge |x-y|_n$), which we denote by $\sigma = \sigma(p_0,\dots,p_k)$, so that $p_0 = x, p_k = y$ and for any $\ell = 0,1,\dots,k-1$, there exists $j \in \{1,2,\dots,2d\}$ such that

$$p_\ell = p_{\ell-1} + \frac{1}{n}\alpha_j.$$

Fix $M_0 \ge 1$ and let $\mathscr{P}(x,y)$ be a family of nearest neighbor paths $\sigma = \sigma(p_0,\dots,p_k)$ from x to y that satisfy $k \le M_0|x-y|_n$. For $\sigma \in \mathscr{P}(x,y)$, define a function D_σ defined on $\mathscr{S}_n \times \mathscr{S}_n$ as follows:

$$D_\sigma(w,z) := \begin{cases} 1, & \text{if there exists } \ell \text{ such that } w = p_\ell \text{ and } z = p_{\ell+1}, \\ 0, & \text{otherwise.} \end{cases}$$

For any function u defined on $\mathscr{S}_n$ and for any $x,y \in \mathscr{S}_n$, we easily see that

$$u(x)-u(y) = \frac{1}{\sharp\mathscr{P}(x,y)} \sum_{\sigma\in\mathscr{P}(x,y)} \sum_{z,w\in\mathscr{S}_n} D_\sigma(w,z)\big(u(w)-u(z)\big).$$

Now let

$$P^{x,y}(w,z) = \frac{1}{\sharp\mathscr{P}(x,y)} \sum_{\sigma\in\mathscr{P}(x,y)} D_\sigma(w,z).$$

For $h \in \mathbb{R}$, $x \in \mathbb{R}^d$ and $i = 1,2,\dots,d$, let

$$\nabla_h^i u(x) = \frac{u(x+he_i)-u(x)}{h}.$$

We then have the following. (See [BKU, Lemma 5.1] for the proof.)

Lemma 4.1.

$$u(x)-u(y) = \frac{1}{n}\sum_{i=1}^d \sum_{z\in\mathscr{S}_n} \Big(P^{x,y}(z+e_i/n,z) - P^{x,y}(z,z+e_i/n)\Big)\nabla_{1/n}^i u(z).$$

Remark 4.2. Let $x = (x_1,\cdots,x_d)$ and $y = (y_1,\cdots,y_d)$ be elements in $\mathscr{S}_n$. Below are some examples of $\mathscr{P}(x,y)$:

(i) Let L_{xy} be the union of the line segment from x to $(y_1,x_2,\cdots,x_d)$, the line segment from $(y_1,x_2,\cdots,x_d)$ to $(y_1,y_2,x_3\cdots,x_d)$, $\cdots$, and the line segment from $(y_1,\cdots,y_{d-1},x_d)$ to y. Set $\mathscr{P}(x,y) = \{L_{xy}\}$ so that $\sharp\mathscr{P}(x,y) = 1$. This was used in [BK], and we do use this in the next section.

(ii) Set $\mathscr{P}(x,y)$ be the set of nearest neighbor paths from x to y such that $k = |x-y|_n$ for each $\sigma = \sigma(p_0,\dots,p_k) \in \mathscr{P}(x,y)$. In this case

$$\sharp\mathscr{P}(x,y) = \frac{(|x-y|_n)!}{\big(n|x_1-y_1|\big)!\big(n|x_2-y_2|\big)!\cdots\big(n|x_d-y_d|\big)!}.$$

(iii) Let $H(x,y)$ be the d-dimensional cube whose vertices consist of $\{(z_1,\cdots,z_d) : z_i \text{ is either } x_i \text{ or } y_i \text{ for } i=1,\cdots,d\}$. Let $\mathscr{P}(x,y)$ be the set of nearest neighbor paths from x to y that consist of a union of the edges of $H(x,y)$. In this case $\sharp\mathscr{P}(x,y) = d\,!$.

Using Lemma 4.1, we can write $\mathscr{E}^n(u,v)$ as follows:
(4.1)

$$\begin{aligned}\mathscr{E}^n(u,v) &= n^{2-d}\sum_{x\in\mathscr{S}_n}\sum_{\substack{\gamma\in\Gamma_n\\ \gamma\ni x}}\frac{\alpha(\gamma)}{l(\gamma)}\sum_{\substack{y\in\mathscr{S}_n\\ y\in\gamma}}(u(x)-u(x_+))(v(x)-v(y))\\ &= n^{-d}\sum_{x\in\mathscr{S}_n}\sum_{\substack{\gamma\in\Gamma_n\\ \gamma\ni x}}\frac{\alpha(\gamma)}{l(\gamma)}\sum_{\substack{y\in\mathscr{S}_n\\ y\in\gamma}}\sum_{i,j=1}^{d}\sum_{z,w\in\mathscr{S}_n}\Big(P^{x,x_+}(z+e_i/n,z)-P^{x,x_+}(z,z+e_i/n)\Big)\\ &\qquad\times\Big(P^{x,y}(w+e_j/n,w)-P^{x,y}(w,w+e_j/n)\Big)\nabla^i_{1/n}u(z)\nabla^j_{1/n}v(w).\end{aligned}$$

For $i,j=1,2,\dots,d$ and $w,z\in\mathscr{S}_n$, set

$$\begin{aligned}G^n_{ij}(z,w) &:= \sum_{x\in\mathscr{S}_n}\sum_{\substack{\gamma\in\Gamma_n\\ \gamma\ni x}}\frac{\alpha(\gamma)}{l(\gamma)}\sum_{\substack{y\in\mathscr{S}_n\\ y\in\gamma}}\Big(P^{x,x_+}(z+e_i/n,z)-P^{x,x_+}(z,z+e_i/n)\Big)\\ &\qquad\times\Big(P^{x,y}(w+e_j/n,w)-P^{x,y}(w,w+e_j/n)\Big);\end{aligned}\tag{4.2}$$

then we see that

$$\mathscr{E}^n(u,v) = \frac{1}{n^d}\sum_{i,j=1}^{d}\sum_{w,z\in\mathscr{S}_n}\nabla^i_{1/n}u(z)\nabla^j_{1/n}v(w)G^n_{ij}(z,w).\tag{4.3}$$

For $i,j=1,2,\dots,d$ and $z\in\mathscr{S}_n$, let

$$\begin{aligned}F^n_{ij}(z) &:= \sum_{w\in\mathscr{S}_n}G^n_{ij}(z,w)\\ &= \sum_{x\in\mathscr{S}_n}\sum_{\substack{\gamma\in\Gamma_n\\ \gamma\ni x}}\frac{\alpha(\gamma)}{l(\gamma)}\Big(P^{x,x_+}(z+e_i/n,z)-P^{x,x_+}(z,z+e_i/n)\Big)\sum_{\substack{y\in\mathscr{S}_n\\ y\in\gamma}}n(x_j-y_j),\end{aligned}\tag{4.4}$$

where x_j, y_j are the j-th coordinate of x,y respectively. Here the equality in (4.4) is because

$$\sum_{w\in\mathscr{S}_n}\Big(P^{x,y}(w+e_j/n,w)-P^{x,y}(w,w+e_j/n)\Big) = n(x_j-y_j).$$

Note that by Assumption 2.1, F^n_{ij} is uniformly bounded, i.e. $\sup_{i,j,n}||F^n_{ij}||_\infty < \infty$.

From now on, we extend the conductances $G^n_{ij}(x,y)$ to $\mathbb{R}^d\times\mathbb{R}^d$ as follows:

$$G^n_{ij}(x,y)=G^n_{ij}([x]_n,[y]_n)\quad \text{for } x,y\in\mathbb{R}^d.$$

We extend $F^n_{ij}(\cdot)$ to $\mathbb{R}^d$ similarly.

We now give an additional assumption needed to obtain weak convergence of the processes.

Assumption 4.3. *i) There exist a matrix-valued functions $a(x)=(a_{ij}(x))$ on $\mathbb{R}^d$ (which is non-symmetric in general) so that for any $i,j=1,2,\dots,d$, the functions $F^n_{ij}(x)$ converge to $a_{ij}(x)$ locally in $L^1(\mathbb{R}^d)$.*
ii) There exists a Borel measure ν on $\mathbb{R}^d$ such that ν^n converges vaguely to ν as $n\to\infty$.

Remark 4.4. 1) Saying that the F^n_{ij} converge locally in $L^1(\mathbb{R}^d)$ means that for every compact set B,

$$\|F^n_{ij}-a_{ij}\|_B:=\int_B|F^n_{ij}(x)-a_{ij}(x)|\,dx\to 0.$$

Since the F^n_{ij} are uniformly bounded, the convergence locally in $L^1(\mathbb{R}^d)$ is equivalent to the convergence in measure on each compact set. In particular, a subsequence will converge almost everywhere.
2) One may consider the weaker condition that the F^n_{ij} are uniformly bounded and converge to a_{ij} weakly. However, this condition is not sufficient for Theorem 4.6 to hold (see the example at the end of the introduction in [SZ]).
3) By Assumptions 2.1 and 2.3, we can easily see that there exists $\lambda>0$ such that

$$\lambda^{-1}|\xi|^2\le\sum_{i,j=1}^d\xi_i\xi_ja_{ij}(x)\le\lambda|\xi|^2,\quad x,\xi\in\mathbb{R}^d.$$

4) Note that by (2.3), $\{\nu^n\}_n$ is tight and there is a convergent subsequence even without the above assumption. Also, the limiting measure ν is absolutely continuous with respect to the Lebesgue measure on $\mathbb{R}^d$. Further, it holds that $M_4\int_A f^2dx\le\int_A f^2d\nu\le M_5\int_A f^2dx$ for all Borel subset A of $\mathbb{R}^d$, in particular $L^2(\mathbb{R}^d,dx)=L^2(\mathbb{R}^d,\nu)$.

Since a is uniformly elliptic, if we define

$$\mathscr{E}(f,g):=\int_{\mathbb{R}^d}\nabla f(x)\cdot a(x)\nabla g(x)dx$$

then $(\mathscr{E},C^1_c(\mathbb{R}^d))$ is a closable Markovian form on $L^2(\mathbb{R}^d,dx)$; cf. [MR] page 49 etc.. Denote the closure by $(\mathscr{E},\mathscr{F})$. Then we have the following.

Lemma 4.5. *Let $W^{1,2}(\mathbb{R}^d):=\{f\in L^2(\mathbb{R}^d,dx):\nabla f\in L^2(\mathbb{R}^d,dx)\}$. Then,*

$$\{f\in L^2(\mathbb{R}^d,dx):\mathscr{E}(f,f)<\infty\}=W^{1,2}(\mathbb{R}^d)=\mathscr{F}. \tag{4.5}$$

Moreover, if $\mathscr{F}'$ is a subset of $L^2(\mathbb{R}^d,dx)$ such that $(\mathscr{E},\mathscr{F}')$ is a regular Dirichlet form on $L^2(\mathbb{R}^d,dx)$, then $\mathscr{F}'=W^{1,2}(\mathbb{R}^d)$.

PROOF. Because $\mathscr{E}(\cdot,\cdot)$ is comparable to the Dirichlet integral $\|(\nabla\cdot)\|_2^2$, the first equality of (4.5) is clear. Now suppose $(\mathscr{E},\mathscr{F}')$ is a regular Dirichlet form on $L^2(\mathbb{R}^d,dx)$; then since $W^{1,2}(\mathbb{R}^d)$ is the maximal domain (due to the first equality in (4.5)), we have $\mathscr{F}'\subset W^{1,2}(\mathbb{R}^d)$. Using the comparability of $\mathscr{E}(\cdot,\cdot)$ and $\|(\nabla\cdot)\|_2^2$ again, $(\|\nabla\cdot\|_2^2,\mathscr{F}')$ is a regular Dirichlet form. This implies $\mathscr{F}'=W^{1,2}(\mathbb{R}^d)$ (so $W^{1,2}(\mathbb{R}^d)=\mathscr{F}$ as well) and the proof is complete. □

Under the above set-up we have the following, which is the main theorem of this paper.

Theorem 4.6. *Suppose Assumptions 2.1, 2.3 and 4.3 hold. Then for each x and each t_0 the $\mathbb{P}^{[x]_n}$-laws of $\{Y_t^{(n)};0\le t\le t_0\}$ converge weakly with respect to the topology of the space $D([0,t_0],\mathbb{R}^d)$. If Z_t is the canonical process on $D([0,t_0),\mathbb{R}^d)$ and $\mathbb{P}^x$ is the weak limit of the $\mathbb{P}^{[x]_n}$-laws of $Y^{(n)}$, then the process $\{Z_t,\mathbb{P}^x\}$ is the Markov process corresponding to the Dirichlet form $\mathscr{E}$ with domain $W^{1,2}(\mathbb{R}^d)$ with the based measure ν.*

We also have the following local limit theorem.

Theorem 4.7. *Suppose Assumptions 2.1, 2.3 and 4.3 hold. Then the limiting process $\{Z_t,\mathbb{P}^x\}$ in Theorem 4.6 enjoys the jointly locally Hölder continuous heat kernel $p(t,x,y)$ for $(t,x,y)\in(0,\infty)\times\mathbb{R}^d\times\mathbb{R}^d$. Further, the following holds for each $T>0$:*

$$\lim_{n\to\infty}\sup_{x,y\in\mathbb{R}^d}\sup_{t\ge T}|p^n(t,[x]_n,[y]_n)-p(t,x,y)|=0, \tag{4.6}$$

where $[x]_n=([nx_1]/n,[nx_2]/n,\cdots,[nx_d]/n)$.

PROOF. Given the a priori estimates of $p(\cdot,\cdot,\cdot)$ and Theorem 4.6, the proof is standard. So, we only give a sketch. By (3.3), (3.24) and by the Ascoli-Arzelà theorem, there exists a subsequence of $p^n(\cdot,[\cdot]_n,[\cdot]_n)$ which converges uniformly to a jointly continuous function $p(\cdot,\cdot,\cdot)$, say. Thanks to Theorem 4.6, it can be easily checked that $p(\cdot,\cdot,\cdot)$ is the heat kernel of the limiting process. Since the limiting process is unique, we can prove the convergence of the full sequence of $p^n(\cdot,[\cdot]_n,[\cdot]_n)$. The uniform convergence in (4.6) is again a consequence of (3.24). □

To prove Theorem 4.6, we first extend $\mathscr{E}^n$ and define a quadratic form on $L^2(\mathbb{R}^d,dx)$. Define

$$\mathscr{H}_n:=\Big\{E_nu:\ u\text{ is a function on }\mathscr{S}_n\Big\}\cap L^2(\mathbb{R}^d,dx).$$

For $f=E_nu\in\mathscr{H}_n$, define

$$\bar{\mathscr{E}}^n(f,f)=n^{2+d}\sum_{i,j=1}^d\iint_{x\neq y}\nabla^i_{1/n}u(x)\nabla^j_{1/n}u(y)G^n_{ij}(x,y)dxdy.$$

Then we see

$$\bar{\mathscr{E}}^n(f,f) = n^{2+d}\sum_{i,j=1}^{d}\sum_{w,z\in\mathscr{S}_n}\nabla^i_{1/n}u(z)\nabla^j_{1/n}u(w)G^n_{ij}(z,w)n^{-2d} = \mathscr{E}^n(u,u). \tag{4.7}$$

Before proving Theorem 4.6, we state a proposition showing tightness of the laws of $Y^{(n)}$.

Proposition 4.8. *Suppose Assumptions 2.1 and 2.3 hold and let $\{n_j\}$ be a subsequence. Then there exists a further subsequence $\{n_{j_k}\}$ such that*

(a) For each C^∞ function f on $\mathbb{R}^d$ with compact support, $E_{n_{j_k}}(P_t^{n_{j_k}}R_{n_{j_k}}(f))$ converges uniformly on compact subsets; if we denote the limit by P_tf, then the operator P_t is linear and extends to all continuous functions on $\mathbb{R}^d$ with compact support and is the semigroup of a strong Markov process on $\mathbb{R}^d$.

(b) For each x and each t_0 the $\mathbb{P}^{[x]_{n_{j_k}}}$ law of $\{Y_t^{(n_{j_k})}; 0\le t\le t_0\}$ converges weakly to a probability $\mathbb{P}^x$.

Given Proposition 3.3 and Theorem 3.9, the proof of this proposition is very similar to that of [BK, Proposition 6.2], so we omit it.

In the following, we write for $h_1,h_2:\mathscr{S}_n\to\mathbb{R}$

$$\langle h_1,h_2\rangle_{\nu^n} := \sum_{x\in\mathscr{S}_n}h_1(x)h_2(x)\nu^n_x,\ \langle h_1,h_2\rangle_n = n^{-d}\sum_{x\in\mathscr{S}_n}h_1(x)h_2(x),$$

cf. (2.4), and for $f_1,f_2\in L^2(\mathbb{R}^d,dx)=L^2(\mathbb{R}^d,\nu)$,

$$\langle f_1,f_2\rangle_\nu := \int_{\mathbb{R}^d}f_1(x)f_2(x)d\nu,\ \langle f_1,f_2\rangle := \int_{\mathbb{R}^d}f_1(x)f_2(x)dx.$$

PROOF OF THEOREM 4.6. Let U_n^λ be the λ-resolvent for $Y^{(n)}$; this means that

$$U_n^\lambda h(x) = \mathbb{E}^x\int_0^\infty e^{-\lambda t}h(Y_t^{(n)})\,dt$$

for $x\in\mathscr{S}_n$ and $h:\mathscr{S}_n\to\mathbb{R}$. First, note that any subsequence $\{n_j\}$ has a further subsequence $\{n_{j_k}\}$ such that $U^\lambda_{n_{j_k}}(R_{n_{j_k}}f)$ converges uniformly on compacts whenever $f\in C_c(\mathbb{R}^d)$, that is, when f is continuous with compact support. This can be proved similarly to Proposition 4.8, so we refer the reader to [BK].

Now suppose we have a subsequence $\{n'\}$ such that the $U^\lambda_{n'}(R_{n'}f)$ are equicontinuous and converge uniformly on compacts whenever $f\in C_c(\mathbb{R}^d)$. Fix such an f and let H be the limit of $U^\lambda_{n'}(R_{n'}f)$.

In the following, we drop the primes for legibility. Set $u_n=U_n^\lambda(R_nf)$ for $\lambda>0$. We will prove that

$$H\in W^{1,2}(\mathbb{R}^d) \qquad \text{and} \qquad \mathscr{E}^n(u_n,g)\to\mathscr{E}(H,g),\ \forall g\in C_c^2(\mathbb{R}^d) \tag{4.8}$$

along some subsequence. Once we have (4.8), then

$$\begin{aligned}\mathscr{E}(H,g) &= \lim \mathscr{E}^n(u_n,g) = \lim(\langle f,g\rangle_{\nu^n} - \lambda\langle u_n,g\rangle_{\nu^n})\\ &= \langle f,g\rangle_\nu - \lambda\langle H,g\rangle_\nu,\end{aligned}$$

the limit being taken along the subsequence. By (4.8), $H \in W^{1,2}(\mathbb{R}^d)$, and the equality

$$\mathscr{E}(H,g) = \langle f,g\rangle_\nu - \lambda\langle H,g\rangle_\nu \tag{4.9}$$

holds for all $g \in C_c^2(\mathbb{R}^d)$. By Lemma 4.5, the closure of $C_c^2(\mathbb{R}^d)$ with respect to $(\mathscr{E}(\cdot,\cdot)+\langle\cdot,\cdot\rangle_\nu)^{1/2}$ is equal to $W^{1,2}(\mathbb{R}^d)$, and so (4.9) holds for all $g \in W^{1,2}(\mathbb{R}^d)$. Since $W^{1,2}(\mathbb{R}^d)$ is the maximal domain due to (4.5), this implies that H is the λ-resolvent of f for the process corresponding to $(\mathscr{E}, W^{1,2}(\mathbb{R}^d))$, that is, $H = U^\lambda f$. We can then conclude that the full sequence $U_n^\lambda(R_n f)$ (without the primes) converges to $U^\lambda f$ whenever $f \in C_c(\mathbb{R}^d)$. The assertions about the convergence of $\mathbb{P}^{[x]_n}$ then follow as in the proof of [BK, Proposition 6.2]. The rest of the proof will be devoted to proving (4.8).

Step 1. The first step is to show $H \in W^{1,2}(\mathbb{R}^d)$. This can be proved similarly to Step 1 in the proof of [BKU, Theorem 5.5], so we omit the proof.

Step 2. We will show that for some subsequence $\{n'\}$,

$$\mathscr{E}^{n'}(u_{n'},g) \longrightarrow \int_{\mathbb{R}^d} \nabla H(x)\cdot a(x)\nabla g(x)dx = \mathscr{E}(H,g)$$

for any $g \in C_c^2(\mathbb{R}^d)$. Recall (4.3); since $G_{ij}^n(x,y) = 0$ if $|x-y| > M_*/n$ for some $M_* > 0$ and the w,z are on the nearest neighbor paths in $\mathscr{P}(x,y)$, $\mathscr{P}(x,x_+)$ respectively, it is enough to consider w's only for $|w-z| \le M/n$ for some $M > 0$ in the sum of the right hand side of (4.3). So

$$\begin{aligned}\mathscr{E}^n(u_n,g) &= \frac{1}{n^d}\sum_{i,j=1}^d \sum_{z\in\mathscr{S}_n} \nabla^i_{1/n}u_n(z) \sum_{\substack{w\in\mathscr{S}_n\\ |w-z|\le M/n}} \nabla^j_{1/n}g(w)G^n_{ij}(z,w)\\ &= \frac{1}{n^d}\sum_{i,j=1}^d \sum_{z\in\mathscr{S}_n} \nabla^i_{1/n}u_n(z)\nabla^j_{1/n}g(z) \sum_{\substack{w\in\mathscr{S}_n\\ |w-z|\le M/n}} G^n_{ij}(z,w)\\ &\quad + \frac{1}{n^d}\sum_{i,j=1}^d \sum_{z\in\mathscr{S}_n} \nabla^i_{1/n}u_n(z) \sum_{\substack{w\in\mathscr{S}_n\\ |w-z|\le M/n}} \Big(\nabla^j_{1/n}g(w) - \nabla^j_{1/n}g(z)\Big)G^n_{ij}(z,w)\\ &=: I_1^n + I_2^n.\end{aligned}$$

Let K be the support of $g \in C_c^2(\mathbb{R}^d)$. Since $1/n \le M/n \le 1$ for large n and $|w-z| \le M/n$ in the summation defining I_2^n, the z's must lie in the set $K_1 \cap \mathscr{S}_n$, where

$K_1 = \{x \in \mathbb{R}^d : \ d(K,x) \le 1\}$. By using the mean value theorem for g and the definition of $\nabla^i_{1/n} u_n$, we see that for some $0 < \theta, \tilde{\theta} < 1$ depending on z and w,

$$
\begin{aligned}
|I_2^n| &= \Big| n^{-d} \sum_{i,j=1}^d \sum_{z \in \mathscr{S}_n} \nabla^i_{1/n} u_n(z) \sum_{\substack{w \in K_1 \cap \mathscr{S}_n \\ |w-z| \le M/n}} \Big(\nabla^j_{1/n} g(w) - \nabla^j_{1/n} g(z) \Big) G^n_{ij}(z,w) \Big| \\
&= \Big| n^{1-d} \sum_{i,j=1}^d \sum_{z \in \mathscr{S}_n} \Big(u_n(z+e_i/n) - u_n(z) \Big) \\
&\qquad \times \sum_{\substack{w \in K_1 \cap \mathscr{S}_n \\ |w-z| \le M/n}} \Big(\partial_{x_j} g(w + \theta e_j/n) - \partial_{x_j} g(z + \tilde{\theta} e_j/n) \Big) G^n_{ij}(z,w) \Big| \\
&\le \Big(\sup_{|z-z'| \le 1/n} \big| u_n(z) - u_n(z') \big| \Big) \cdot \sup_j ||\partial^2_{x_j} g||_\infty \times \Big(n^{-d} \sum_{i,j=1}^d \sum_{z \in \mathscr{S}_n} \sum_{\substack{w \in K_1 \cap \mathscr{S}_n \\ |w-z| \le M/n}} \big| G^n_{ij}(z,w) \big| \Big) \\
&=: \Big(\sup_{|z-z'| \le 1/n} \big| u_n(z) - u_n(z') \big| \Big) \cdot \sup_j ||\partial^2_{x_j} g||_\infty \times I_3^n.
\end{aligned}
$$

We now estimate I_3^n. Let $K_2 = \{x \in \mathbb{R}^d : \ d(K_1,x) \le 1\}$. Then,

$$
\begin{aligned}
I_3^n &= n^{-d} \sum_{i,j=1}^d \sum_{z \in \mathscr{S}_n} \sum_{\substack{w \in K_1 \cap \mathscr{S}_n \\ |w-z| \le M/n}} \Big| \sum_{x \in \mathscr{S}_n} \sum_{\substack{\gamma \in \Gamma_n \\ \gamma \ni x}} \frac{\alpha(\gamma)}{l(\gamma)} \sum_{\substack{y \in \mathscr{S}_n \\ y \in \gamma}} \Big(P^{x,x_+}(z+e_i/n, z) - P^{x,x_+}(z, z+e_i/n) \Big) \\
&\qquad \times \Big(P^{x,y}(w+e_j/n, w) - P^{x,y}(w, w+e_j/n) \Big) \Big| \\
&\le n^{-d} \sum_{x \in K_2 \cap \mathscr{S}_n} \sum_{\substack{\gamma \in \Gamma_n \\ \gamma \ni x}} \frac{\alpha(\gamma)}{l(\gamma)} \sum_{\substack{y \in \mathscr{S}_n \\ y \in \gamma}} \Big\{ \sum_{i=1}^d \sum_{z \in \mathscr{S}_n} \Big(P^{x,x_+}(z+e_i/n, z) + P^{x,x_+}(z, z+e_i/n) \Big) \\
&\qquad \times \sum_{j=1}^d \sum_{\substack{w \in K_1 \cap \mathscr{S}_n \\ |w-z| \le M/n}} \Big(P^{x,y}(w+e_j/n, w) + P^{x,y}(w, w+e_j/n) \Big) \Big\}.
\end{aligned}
$$

Note that for $x \in \mathscr{S}_n$ and $\gamma \ni x$, we have $|x - x_+|_1 \le M/n$, $|x-y|_1 \le M/n$ for all $y \in \gamma$, where $|x|_1 := \sum_{i=1}^d |x_i|$. So

$$
\begin{aligned}
&\sum_{j=1}^d \sum_{\substack{w \in K_1 \cap \mathscr{S}_n \\ |w-z| \le M/n}} \Big(P^{x,y}(w+e_j/n, w) + P^{x,y}(w, w+e_j/n) \Big) \\
&\qquad \le \sum_{j=1}^d \sum_{w \in \mathscr{S}_n} \Big(P^{x,y}(w+e_j/n, w) + P^{x,y}(w, w+e_j/n) \Big) \le M_0 n |x-y|_1 \le M_0 M
\end{aligned}
$$

for some $M_0 > 0$ and similarly

$$\sum_{i=1}^{d}\sum_{z\in\mathscr{S}_n}\Big(P^{x,x_+}(z+e_i/n,z)+P^{x,x_+}(z,z+e_i/n)\Big)\le M_0 n|x-x_+|_1\le M_0 M.$$

So we obtain,

$$I_3^n \le n^{-d}\sum_{x\in K_2\cap\mathscr{S}_n}\sum_{\substack{\gamma\in\Gamma_n\\ \gamma\ni x}}\frac{\alpha(\gamma)}{l(\gamma)}\sum_{\substack{y\in\mathscr{S}_n\\ y\in\gamma}}(M_0M)^2\le C\mu^n(K_2).$$

So, I_3^n is uniformly bounded in n and hence I_2^n converges to 0 as n tends to ∞ since the $\{u_n\}$ are equicontinuous.

Finally we consider the term I_1^n:

$$\begin{aligned} I_1^n &= \frac{1}{n^d}\sum_{i,j=1}^{d}\sum_{z\in\mathscr{S}_n}\nabla^i_{1/n}u_n(z)\nabla^j_{1/n}g(z)F^n_{ij}(z)\\ &= \sum_{i,j=1}^{d}\int_{\mathbb{R}^d}\nabla^i_{1/n}E_nu_n(x)\nabla^j_{1/n}E_ng(x)F^n_{ij}(x)dx. \end{aligned}$$

Observe that if f_n converges to f weakly in $L^2(\mathbb{R}^d)$ and g_n converges to g boundedly and almost everywhere, then f_ng_n converges to fg weakly. To see this, if $h\in L^2(\mathbb{R}^d)$,

$$\int (f_ng_n)h-\int (fg)h=\int f_n(g_n-g)h+\Big[\int f_ngh-\int fgh\Big].$$

The term inside the brackets on the right hand side goes to 0 since f_n converges to f weakly and the boundedness of g implies that gh is in $L^2(\mathbb{R}^d)$. The first term on the right hand side is bounded, using Cauchy-Schwarz, by $\|f_n\|_2\,\|(g_n-g)h\|_2$. The factor $\|f_n\|_2$ is uniformly bounded since f_n converges weakly in $L^2(\mathbb{R}^d)$, while $\|(g_n-g)h\|_2$ converges to 0 by dominated convergence.

Since some subsequence of $\nabla^i_{1/n}E_nu_n$ converges to $v_i=\partial_{x_i}H$ weakly in $L^2(\mathbb{R}^d,dx)$ (this can be verified when proving $H\in W^{1,2}(\mathbb{R}^d)$ in Step 1; see the proof of [BKU, Theorem 5.5]), and for some further subsequence F^n_{ij} converges to a_{ij} boundedly and almost everywhere (by Assumption 4.3 and Remark 4.4) and $\nabla^j_{1/n}E_ng$ converges to $\partial_{x_j}g$ uniformly on compact sets (because $g\in C^2_c(\mathbb{R}^d)$), we see that, along this further subsequence, the right hand side goes to

$$\sum_{i,j=1}^{d}\int_{\mathbb{R}^d}\partial_{x_i}H\,\partial_{x_j}g\,a_{ij}\,dx=\int_{\mathbb{R}^d}\nabla H(x)\cdot a(x)\nabla g(x)dx.$$

Hence

$$\mathscr{E}^{n'}(u_{n'},g)\to\mathscr{E}(H,g).$$

This completes the proof of (4.8) and hence the theorem. □

5 Discrete approximation

In this section, we show how the results of the previous sections can be applied to approximate a non-symmetric diffusion in divergence form by a sequence of Markov chains with bounded cycle decomposition. In Section 5.1, we present two special examples which will play the role of building blocks for our construction. In Section 5.2, we apply a two scale methods for the concrete approximation.

5.1 Some computation of $F_{ij}^n(\cdot)$

In this subsection, we compute $F_{ij}^n(\cdot)$ in (4.4) for two particular cases. The computation is useful in the next subsection. First, for each $x \in \mathscr{S}_n$ and $\gamma \ni x$, set

(5.1) $$K(x,\gamma)_j := \sum_{\substack{y \in \mathscr{S}_n \\ y \in \gamma}} n(x_j - y_j).$$

Then, by (4.4) we have,

(5.2) $$F_{ij}^n(z) = \sum_{x \in \mathscr{S}_n} \sum_{\substack{\gamma \in \Gamma_n \\ \gamma \ni x}} \frac{\alpha(\gamma)}{l(\gamma)} \Big(P^{x,x_+}(z + e_i/n, z) - P^{x,x_+}(z, z + e_i/n) \Big) K(x,\gamma)_j$$

for $z \in \mathscr{S}_n,\ i,j = 1,2,\ldots,d$.

From now on, for each $x,y \in \mathscr{S}_n$, we choose $\mathscr{P}(x,y)$ as in Remark 4.2 (i), namely we set $\mathscr{P}(x,y) = \{L_{xy}\}$ where L_{xy} is the union of the line segments from x to y mentioned in Remark 4.2 (i). We now consider two concrete choices of Γ_n.

Example 1: Let Λ_n be a subset of unordered pair $\{x,y\}, x \neq y$ of $\mathscr{S}_n$ and let $\Gamma_n = \{\gamma_{xy} = (x,y,x) : \{x,y\} \in \Lambda_n\}$, where γ_{xy} is a cycle of length 2 that consists of x and y. Note that $(x,y,x) = (y,x,y)$ and $\gamma_{xy} = \gamma_{yx}$. For simplicity we write

$$\alpha(\gamma_{xy}) = \alpha(x,y) = \alpha(y,x), \quad \{x,y\} \in \Lambda_n.$$

We call such a cycle *a two cycles*. In this case, we have $K(x,\gamma_{xy})_j = n(x_j - y_j)$, so that (5.2) can be written as

(5.3) $$F_{ij}^n(z) = \sum_{\gamma_{xy} \in \Gamma_n} \alpha(x,y) \Big(P^{x,y}(z + e_i/n, z) - P^{x,y}(z, z + e_i/n) \Big) n(x_j - y_j).$$

Now for $x,y \in \mathscr{S}_n$ satisfying $x_i \le y_i$, set $M(x,y,i) = \{z \in \mathscr{S}_n : z_k = y_k$ for all $k < i, x_i \le z_i \le y_i - 1/n, z_{k'} = x_{k'}$ for all $k' > i\}$. Note that $\sharp M(x,y,i) = n(y_i - x_i)$. Then, we can easily see

$$P^{x,y}(z + e_i/n, z) - P^{x,y}(z, z + e_i/n) = 0 - 1_{M(x,y,i)}(z) = -1_{M(x,y,i)}(z),$$

for all $x,y \in \mathscr{S}_n$ with $x_i \le y_i$. Plugging this into (5.3), we obtain

(5.4) $$F_{ij}^n(z) = \sum_{\substack{\gamma_{xy} \in \Gamma_n \\ x_i \le y_i}} \alpha(x,y) n(y_j - x_j) 1_{M(x,y,i)}(z).$$

In particular let us consider the collection $\Lambda_n = \{\{w, w+V(w)/n\}, w \in \mathscr{S}_n\}$ where $V : \mathscr{S}_n \longrightarrow \mathbb{Z}^d \setminus \{0\}$, $V(z) = (V_1(z), \cdots, V_d(z))$, is a given map of bounded range L, that is

$$L = L(V) = \max_{i=1}^{d} \sup_{z \in \mathscr{S}_n} |V_i(z)| < \infty \tag{5.5}$$

and we set

$$\alpha(\gamma_{w,w+V(w)/n}) =: \alpha(w).$$

Note that there is no ambiguity for this notation because by definition of Λ_n, for each $w \in \mathscr{S}_n$ we can naturally choose one element of Γ_n in this case.

For fixed $z \in \mathscr{S}_n$ and $i = 1, ..., d$, let $N^+(z,i) = \{w \in \mathscr{S}_n : V_i(w) > 0, w_k + V_k(w)/n = z_k,$ for all $k < i, w_i \le z_i \le w_i + V_i(w)/n - 1/n, w_{k'} = z_{k'},$ for all $k' > i\}$ and $N^-(z,i) = \{w \in \mathscr{S}_n : V_i(w) < 0, w_k = z_k,$ for all $k < i, w_i + V_i(w)/n \le z_i \le w_i - 1/n, w_{k'} + V_{k'}(W)/n = z_{k'},$ for all $k' > i\}$. Note that in view of (5.5) we see that

$$N^+(z,i), N^-(z,i) \subset D_n(z,L) := \{w \in \mathscr{S}_n : \max_{i=1}^{d} |z_i - w_i| \le L/n.\} \tag{5.6}$$

In particular

$$0 \le \sharp N^+(z,i), \sharp N^-(z,i) \le \sharp D_n(z,L) \le CL^d. \tag{5.7}$$

The computation of F^n_{ij}, yields

$$F^n_{ij}(z) = \Big(\sum_{w \in N^+(z,i)} \alpha(w) V_j(w) - \sum_{w \in N^-(z,i)} \alpha(w) V_j(w) \Big). \tag{5.8}$$

Set

$$\bar{F}^n_{ij}(z) = \alpha(z) V_i(z) \cdot V_j(z). \tag{5.9}$$

Then

$$F^n_{ij}(z) = \bar{F}^n_{ij}(z) + R^{1,n}_{ij}(z) + R^{2,n}_{ij}(z) + R^{3,n}_{ij}(z), \tag{5.10}$$

where

$$R^{1,n}_{ij}(z) = \sum_{w \in N^+(z,i)} \big(\alpha(w) - \alpha(z)\big) V_j(w) - \sum_{w \in N^-(z,i)} \big(\alpha(w) - \alpha(z)\big) V_j(w), \tag{5.11}$$

$$R^{2,n}_{ij}(z) = \alpha(z) \Big(\sum_{w \in N^+(z,i)} \big(V_j(w) - V_j(z)\big) - \sum_{w \in N^-(Z,i)} \big(V_j(w) - V_j(z)\big) \Big), \tag{5.12}$$

and

$$R^{3,n}_{ij}(z) = \alpha(z) V_j(z) \Big(\sharp N^+(z,i) - \sharp N^-(z,i) - V_i(z) \Big). \tag{5.13}$$

This simplifies greatly if the vector $V(\cdot)$ is a $(2L/n)$-piecewise constant function, that is $V(x) = V(2L[\frac{x}{2L}]_n)$ for $x \in \mathscr{S}_n$ (in other words, $V(\cdot)$ is constant inside each cell of $(2L)\mathscr{S}_n$). Let $z \in \mathscr{S}_n$ be such that

$$V(w) = V(z), \qquad \forall w \in D_n(z,L). \tag{5.14}$$

Then we simply have

$$N^+(z,i) = \{V_i(z) > 0\} \cap \{w = (z_1 + V_1(z)/n, ..., z_{i-1} + V_{i-1}(z)/n, w_i, z_{i+1}, ..., z_d) \tag{5.15}$$
$$: w_i \le z_i \le w_i + V_i(z)/n - 1/n\},$$

and

$$N^-(z,i) = \{V_i(z) < 0\} \cap \{w = (z_1, ..., z_{i-1}, w_i, z_{i+1} + V_{i+1}(z)/n, ..., z_d + V_d(z)/n) \tag{5.16}$$
$$: w_i + V_i(z)/n \le z_i \le w_i - 1/n\},$$

and thus

$$\sharp N^+(z,i) = V_i(z) \cdot 1_{\{V_i(z)>0\}}, \qquad \sharp N^-(z,i) = -V_i(z) \cdot 1_{\{V_i(z)<0\}}.$$

We see that

$$R^{2,n}_{ij}(z) = R^{3,n}_{ij}(z) = 0, \tag{5.17}$$

and

$$R^{1,n}_{ij}(z) = \Big(\sum_{w \in N^+(z,i)} (\alpha(w) - \alpha(z)) - \sum_{w \in N^-(z,i)} (\alpha(w) - \alpha(z)) \Big) \cdot V_j(z). \tag{5.18}$$

Example 2: Fix $N \ge 1$ and $1 \le l \ne m \le d$, and for given $x \in \mathscr{S}_n$ let $\gamma^{(l,m)}_{N,x}$ be a cycle of length $8N$ with range 1 (that is to nearest neighbors) that makes a regular square with vertices

$$x, x - \frac{2N}{n} e_l, x - \frac{2N}{n} e_l - \frac{2N}{n} e_m, x - \frac{2N}{n} e_m.$$

We call such a cycle *a rotational cycle of length* $8N$. Note that $\gamma^{(m,l)}_{N,x}$ passes through the same points, but with a different orientation.

Let us consider the family of cycles

$$\Gamma_n = \{\gamma^{(l,m)}_{N,x} : x \in \mathscr{S}_n\}$$

with weights

$$\alpha(\gamma^{(l,m)}_{N,x}) =: \alpha(x), \quad x \in \mathscr{S}_n.$$

As in Example 1, there is no ambiguity for this notation because by definition of Λ_n, for each $x \in \mathscr{S}_n$ we can naturally choose one element of Γ_n in this case. In this case, $P^{x,x_+}(z + e_k/n, z) - P^{x,x_+}(z, z + e_k/n)$ is 1 if $x = z + e_k/n$, $x_+ = z$, it is -1 if

$x=z$, $x_+=z+e_k/n$, and 0 otherwise. Plugging this into (5.2), we have

$$F^n_{ij}(z)=\sum_{\substack{\gamma\in\Gamma_n\\ \gamma\ni z}}\frac{\alpha(\gamma)}{8N}(-\delta_{z_+,z+\frac{1}{n}e_i})K(z,\gamma)_j+\sum_{\substack{\gamma\in\Gamma_n\\ \gamma\ni z+e_i/n}}\frac{\alpha(\gamma)}{8N}\delta_{(z+\frac{1}{n}e_i)_+,z}K(z+\frac{1}{n}e_i,\gamma)_j$$

$$=\frac{1}{8N}\Big\{-\sum_{k=1}^{2N}\alpha(z+\frac{2N}{n}e_m+\frac{k}{n}e_l)K(z,\gamma^{(l,m)}_{N,z+\frac{2N}{n}e_m+\frac{k}{n}e_l})_j\cdot\delta_{il}$$
$$-\sum_{k=1}^{2N}\alpha(z+\frac{k}{n}e_m)K(z,\gamma^{(l,m)}_{N,z+\frac{k}{n}e_m})_j\cdot\delta_{im}+\sum_{k=1}^{2N}\alpha(z+\frac{k}{n}e_l)K(z+\frac{1}{n}e_l,\gamma^{(l,m)}_{N,z+\frac{k}{n}e_l})_j\cdot\delta_{il}$$
$$+\sum_{k=1}^{2N}\alpha(z+\frac{2N}{n}e_l+\frac{k}{n}e_m)K(z+\frac{1}{n}e_m,\gamma^{(l,m)}_{N,z+\frac{2N}{n}e_l+\frac{k}{n}e_m})_j\cdot\delta_{im}\Big\}. \tag{5.19}$$

By definition, we have

$$K(z+\frac{1}{n}e_l,\gamma^{(l,m)}_{N,z+\frac{k}{n}e_l})_j=K(z,\gamma^{(l,m)}_{N,z+\frac{k-1}{n}e_l})_j \quad\text{and}$$
$$K(z+\frac{1}{n}e_m,\gamma^{(l,m)}_{N,z+\frac{2N}{n}e_l+\frac{k}{n}e_m})_j=K(z,\gamma^{(l,m)}_{N,z+\frac{2N}{n}e_l+\frac{k-1}{n}e_m})_j.$$

Now we compute $K(z,\gamma^{(l,m)}_{N,z+\frac{k}{n}e_p})_j$ for $p\in\{l,m\}$. First, if $p\neq j$, then $K(z,\gamma^{(l,m)}_{N,z+\frac{k}{n}e_p})_j=0$ unless $\{p,j\}=\{l,m\}$. A simple computation gives

$$K(z,\gamma^{(l,m)}_{N,z+\frac{k}{n}e_p})_j=\{2\sum_{s=1}^{2N}s+2N(2N-1)\}\delta_{\{p,j\},\{l,m\}}=\delta_{\{p,j\},\{l,m\}}8N^2.$$

Second, if $p=j$, then

$$K(z,\gamma^{(l,m)}_{N,z+\frac{k}{n}e_p})_j=-2\sum_{s=1}^{k}s-k(2N-1)+2\sum_{s=1}^{2N-k}s+(2N-k)(2N-1)=8N(N-k).$$

Similarly, for p,q such that $\{p,q\}=\{l,m\}$, we have

$$K(z,\gamma^{(l,m)}_{N,z+\frac{2N}{n}e_q+\frac{k}{n}e_p})_j=\begin{cases}-\delta_{\{p,q\},\{l,m\}}8N^2 & \text{if } p\neq j,\\ 8N(N-k) & \text{if } p=j.\end{cases}$$

Putting these into (5.19), we have for all $z \in \mathscr{S}_n$, $F^n_{ij}(z) = 0$, $i, j \notin \{l, m\}$ and

$$\begin{aligned}
F^n_{lm}(z) &= N\sum_{k=1}^{2N}\Big(\alpha(z+\frac{2N}{n}e_m+\frac{k}{n}e_l)+\alpha(z+\frac{k}{n}e_l)\Big),\\
F^n_{ml}(z) &= -N\sum_{k=1}^{2N}\Big(\alpha(z+\frac{2N}{n}e_l+\frac{k}{n}e_m)+\alpha(z+\frac{k}{n}e_m)\Big),\\
F^n_{ll}(z) &= \Big(\alpha(z+\frac{1}{n}e_l)+\alpha(z+\frac{2N}{n}e_m+\frac{2N}{n}e_l)\Big)N\\
&\quad+\sum_{k=1}^{2N-1}\Big(\alpha(z+\frac{k+1}{n}e_l)-\alpha(z+\frac{2N}{n}e_m+\frac{k}{n}e_l)\Big)(N-k),\\
F^n_{mm}(z) &= \Big(\alpha(z+\frac{2N}{n}e_m)+\alpha(z+\frac{2N}{n}e_l+\frac{1}{n}e_m)\Big)N\\
&\quad+\sum_{k=1}^{2N-1}\Big(\alpha(z+\frac{2N}{n}e_l+\frac{k+1}{n}e_m)-\alpha(z+\frac{k}{n}e_m)\Big)(N-k).
\end{aligned}$$

As above set

$$\bar{F}^n_{lm}(z) = -\bar{F}^n_{lm}(z) = 4N^2\alpha(z),\ \bar{F}^n_{ll}(z) = \bar{F}^n_{mm}(z) = 2N\alpha(z) \tag{5.20}$$

then

$$F^n_{ij}(z) = \bar{F}^n_{ij}(z) + R^{4,n}_{ij}(z) \tag{5.21}$$

where

$$\begin{aligned}
R^{4,n}_{lm}(z) &= N\sum_{k=1}^{2N}\Big(\alpha(z+\frac{2N}{n}e_m+\frac{k}{n}e_l)+\alpha(z+\frac{k}{n}e_l)-2\alpha(z)\Big),\\
R^{4,n}_{ml}(z) &= -N\sum_{k=1}^{2N}\Big(\alpha(z+\frac{2N}{n}e_l+\frac{k}{n}e_m)+\alpha(z+\frac{k}{n}e_m)-2\alpha(z)\Big),\\
R^{4,n}_{ll}(z) &= \Big(\alpha(z+\frac{1}{n}e_l)+\alpha(z+\frac{2N}{n}e_m+\frac{2N}{n}e_l)-2\alpha(z)\Big)N\\
&\quad+\sum_{k=1}^{2N-1}\Big(\alpha(z+\frac{k+1}{n}e_l)-\alpha(z+\frac{2N}{n}e_m+\frac{k}{n}e_l)\Big)(N-k),\\
R^{4,n}_{mm}(z) &= \Big(\alpha(z+\frac{2N}{n}e_m)+\alpha(z+\frac{2N}{n}e_l+\frac{1}{n}e_m)-2\alpha(z)\Big)N\\
&\quad+\sum_{k=1}^{2N-1}\Big(\alpha(z+\frac{2N}{n}e_l+\frac{k+1}{n}e_m)-\alpha(z+\frac{k}{n}e_m)\Big)(N-k).
\end{aligned}$$

5.2 Discrete approximation

In this last subsection, we give a concrete approximation of non-symmetric diffusions in divergence form.

For a matrix $a = (a_{ij})_{i,j=1}^d$ we denote by $\tilde{a}$ the symmetric and by $\hat{a}$ the antisymmetric part. Also for $\xi \in \mathbb{R}^d$, let

$$\langle \xi, a\xi \rangle = \sum_{i,j}^d \xi_i a_{ij} \xi_j = \sum_{i,j}^d \xi_i \tilde{a}_{ij} \xi_j \text{ and } \|a\| := \|a\|_{\infty \to \infty} = \max_i \sum_j |a_{ij}|.$$

For $\varepsilon, M_1, M_2 > 0$ be denote by

$$\mathscr{M}_d(\varepsilon, M_1.M_2) = \{a = \tilde{a} + \hat{a} : \langle \xi, a\xi \rangle \geq \varepsilon \langle \xi, \xi \rangle^2 \quad \text{and} \quad \|\tilde{a}\| \leq M_1, \ \|\hat{a}\| \leq M_2\}$$

the set of uniformly elliptic, bounded matrices. Clearly a is symmetric if and only if $M_2 = 0$.

Given a measurable map $a : \mathbb{R}^d \to \mathscr{M}_d(\varepsilon, M_1, M_2)$, our goal is to find a sequence of Markov chains that approximate the diffusion process whose divergence form is determined by a. Thanks to Theorem 4.6, all we need is to find a sequence (Γ_n, α_n) where Γ_n is a collection of cycles $\gamma_i^n, i \in I$ in $\mathscr{S}_n$ with weights $\alpha_n(\gamma_i^n) \geq 0$ such that (2.2), (2.8) and (2.9) are satisfied and the corresponding $F_{ij}^n(\cdot)$ converges locally in $L^1(\mathbb{R}^d)$ to a_{ij}, that is, for all K compact subset of $\mathbb{R}^d$

$$\lim_{n\to\infty} \|F_{ij}^n - a_{ij}\|_K = \lim_{n\to\infty} \int_K |F_{ij}^n(x) - a_{ij}(x)|\, dx = 0, \quad \forall i, j = 1, .., d, \tag{5.22}$$

where as usual we write $F_{ij}^n(x) = F_{ij}^n([x]_n)$ for $x \in \mathbb{R}^d$. In Theorem 5.5, which is our main theorem, we will prove that it is possible to find such a sequence.

Clearly, if $a^n : \mathbb{R}^d \to \mathscr{M}_d(\varepsilon, M_1, M_2)$ is a sequence such that

$$\lim_{n\to\infty} \|a_{ij}^n - a_{ij}\|_K = 0 \qquad \text{and} \qquad \lim_{n\to\infty} \|F_{ij}^n - a_{ij}^n\|_K = 0, \tag{5.23}$$

then by the triangle inequality (5.22) holds.

Our construction will be based on a two scale procedure: we will discretize the matrix $a(\cdot)$ at intermediate scale $r_n = [n^{1-\beta}]/n$, for some $\beta \in (0,1)$ and then construct the corresponding chain on $\mathscr{S}_n$ at microscopic scale $1/n$.

More precisely, take $\beta \in (0,1)$, set $r_n = [n^{1-\beta}]/n \in (1/n)\mathbb{Z}_+$, $J_{r_n} = r_n\mathbb{Z}^d \subset \mathscr{S}_n$, and let

$$Q(x, r_n) = \{y \in \mathbb{R}^d : 0 \leq \min_i (y_i - x_i) \leq \max_i (y_i - x_i) < r_n\}, \qquad x \in J_{r_n},$$

be a partition of disjoint cubes of $\mathbb{R}^d$. For a measurable map $a : \mathbb{R}^d \to \mathscr{M}_d(\varepsilon, M_1, M_2)$, set

$$a_{ij}^n(y) = \sum_{x \in J_{r_n}} \Big(\frac{1}{r_n^d} \int_{Q(x,r_n)} a_{ij}(z)\, dz \Big) 1_{Q(x,r_n)}(y), \quad y \in \mathbb{R}^d. \tag{5.24}$$

Then

$$a^n(x) \in \mathscr{M}_d(\varepsilon, M_1, M_2), \quad \forall x \in \mathscr{S}_n,$$

and for every compact $K \subset \mathbb{R}^d$ we have

$$\lim_{n\to\infty} \|a_{ij}^n - a_{ij}\|_K = 0. \tag{5.25}$$

Furthermore, $a^n_{ij}(x)$ is a r_n-piecewise constant function, that is

$$a^n_{ij}(x) = a^n_{ij}\Big((r_n[\frac{x_1}{r_n}], \cdots, r_n[\frac{x_d}{r_n}])\Big),\ x = (x_1, \cdots x_d) \in \mathscr{S}_n. \tag{5.26}$$

As we see, r_n is an "intermediate" scale and a^n_{ij} is an approximation of a_{ij} which is constant in each cell of J_{r_n}.

We start with a trivial observation: let (Γ^1_n, α^1_n) and (Γ^2_n, α^2_n) be two such collections and consider the merged collection $\Gamma_n = \Gamma^1_n \cup \Gamma^2_n$ with weights $\alpha_n(\gamma_n) = \alpha^i_n(\gamma_n)$ if $\gamma_n \in \Gamma^i_n$, then the corresponding F^n satisfies the *additive rule*

$$F^n_{ij} = F^{1,n}_{ij} + F^{2,n}_{ij}. \tag{5.27}$$

Also if both (Γ^i_n, α^i_n) satisfy (2.2) then of course (Γ_n, α_n) satisfy (2.2).

This additive rule will be a very useful tool for our construction and we will proceed iteratively. We will be dealing with the symmetric part $\tilde{a}(x)$ and anti-symmetric part $\hat{a}(x)$ separately. We will need to represent the symmetric part in diagonalized form, that is

$$\big\langle \xi, \tilde{a}(x)\xi \big\rangle = \sum_{k=1}^{d} \nu_k(x) \big\langle \xi, V^k(x) \big\rangle^2, \tag{5.28}$$

where $\nu_k(x) \geq 0$ and $V^k(x) \in \mathbb{R}^d$. Of course both coefficients $\nu_k(x)$ and vectors $V^k(x)$ are not unique. In particular one can always replace, for each k, $\nu_k(x)$ by $\nu_k(x)/\alpha^2$ and $V^k(x)$ by $\alpha \cdot V^k(x)$ for any $\alpha > 0$. In particular using rescaling we get a normalized diagonalized form such that

$$|V^k(x)|^2 := \big\langle V^k(x), V^k(x) \big\rangle = 1 \qquad \forall x \in \mathbb{R}^d,\ k = 1, ..., d. \tag{5.29}$$

Usually one chooses $\nu_k(x) = \lambda_k(x)$ to be the eigenvalues and $V^k(x)$ the normalized eigenvectors of the matrix $\tilde{a}(x)$. However this is of course not the only choice to represent the quadratic form in diagonalized form. Denoting by $\lambda_{\max}(\tilde{a}(x))$ the largest eigenvalue of $\tilde{a}(x)$, assuming the normalization (5.29) we see that

$$0 \leq \nu_k(x) \leq \lambda_{\max}(\tilde{a}(x)) \leq \sqrt{d}\|\tilde{a}(x)\| \qquad \forall x \in \mathbb{R}^d,\ k = 1, ..., d. \tag{5.30}$$

This follows from the simple fact that

$$\lambda_{\max}(\tilde{a}(x)) \geq \big\langle V_k(x), \tilde{a}(x) V_k(x) \big\rangle = \sum_{j=1}^{d} \nu_j(x) \big\langle V_j(x), V_k(x) \big\rangle^2 \geq \nu_k(x),$$

and, for positive definite symmetric matrix $\tilde{a}(x)$,

$$\lambda_{\max}(\tilde{a}(x)) = \|\tilde{a}(x)\|_{2\to 2} \leq \sqrt{d}\|\tilde{a}(x)\|.$$

The following iterative procedure based on the Feshbach map, cf. [BFS] shows how to find a representation in diagonalized form, avoiding the computation of the eigenvalues and eigenvectors of $\tilde{a}(x)$.

Take a symmetric positive definite matrix $a \in \mathbb{R}^{d\times d}$ and set

$$\nu = \frac{1}{a_{11}} > 0 \qquad V^t = (a_{11}, a_{12}, ..., a_{1d}) \in \mathbb{R}^d.$$

Next define the symmetric matrix $a' \in \mathbb{R}^{d\times d}$, in terms of the quadratic form

$$\langle \xi, a'\xi \rangle = \langle \xi, a\xi \rangle - \nu \langle \xi, V \rangle^2, \quad \xi \in \mathbb{R}^d, \tag{5.31}$$

that is $a' = a - \nu V \cdot V^t$. Note that the first row and column of the matrix a' vanish:

$$a'_{1i} = a'_{i1} = 0, \qquad i = 1, ..., d.$$

The Feshbach transform of the matrix a is a matrix $F_d(a) \in \mathbb{R}^{(d-1)\times(d-1)}$ and it is given by

$$(F_d(a))_{i-1,j-1} = a'_{ij}, \quad 2 \le i, j \le d.$$

Lemma 5.1. *For a symmetric positive definite matrix $a \in \mathbb{R}^{d\times d}$, let $\lambda_{\min}(a), \lambda_{\max}(a)$ be the smallest and largest eigenvalue of a. Then*

$$\lambda_{\min}(a) \le \lambda_{\min}(F_d(a)) \le \lambda_{\max}(F_d(a)) \le \lambda_{\max}(a).$$

PROOF. Let $\mu \in \mathbb{R}$ be an eigenvalue of a'. Clearly $\mu = 0$ is eigenvalue. We claim that if $\mu \ne 0$ then

$$\lambda_{\min}(a) \le \mu \le \lambda_{\max}(a).$$

This shows the lemma. We may assume that $\mu \ne \lambda_i, i = 1, ..., d$, where $\lambda_1 = \lambda_{\min}(a) \le \lambda_2 \le ... \le \lambda_d = \lambda_{\max}(a)$ are the eigenvalues of the matrix a. Let $S \in \mathbb{R}^{d\times d}$ be the orthogonal matrix such that

$$S \cdot a \cdot S^t = D$$

where D is the diagonal matrix of the eigenvalues of a and set $Z = SV$. Then

$$0 = \det(a' - \mu I) = \det(S(a' - \mu I)S^t) = \det((D - \mu I) - \frac{1}{a_{11}} Z \cdot Z^t).$$

Since $\mu \ne \lambda_i$, $(D - \mu I)^{-1}$ exists and thus

$$f(\mu) := \det(I - (D - \mu I)^{-1} \frac{1}{a_{11}} Z \cdot Z^t) = 1 - \frac{1}{a_{11}} \sum_{i=1}^{d} \frac{Z_i^2}{\lambda_i - \mu} = 0.$$

Here, the first equality is due to the following fact: for d-dimensional vectors U and V, $\det(I - UV^t) = 1 - \langle U, V \rangle$ where UV^t is a $(d \times d)$-matrix. Note that f is monotone decreasing on $(-\infty, \lambda_1) \cup (\lambda_d, \infty)$. Thus, if $\mu \ne 0$ and $f(\mu) = 0$, then $\mu \in (\lambda_1, \lambda_d)$. □

We introduce the following iterative procedure to represent a symmetric matrix $\tilde{a}(x) \in \mathscr{M}_d(\varepsilon, M_1, 0)$ in diagonalized form: We define a sequence of symmetric matrices $a^1(x), ..., a^d(x)$ as follows

$$a^1(x) = \tilde{a}(x)$$

$$\langle \xi, a^{k+1}(x)\xi \rangle = \langle \xi, a^k(x)\xi \rangle - \nu_k(x) \langle V_k(x), \xi \rangle^2, \quad k = 1, ..., d-1,$$

where for $k = 1,...,d$

$$\nu_k(x) = \frac{1}{a^k_{kk}(x)}, \qquad (V_k(x))^t = (0,...,0,a^k_{kk}(x),...,a^k_{kd}(x)) \in \mathbb{R}^d.$$

For example when $d = 2$, we simply have

$$\nu_1(x) = \frac{1}{a_{11}(x)}, \qquad V_1(x) = (a_{11}(x), a_{12}(x))$$

and

$$\nu_2(x) = \frac{1}{a_{22}(x) - \frac{(a_{12}(x))^2}{a_{11}(x)}}, \quad V_2(x) = (0, a_{22}(x) - \frac{(a_{12}(x))^2}{a_{11}(x)}).$$

By construction we have that

$$a^k_{ij}(x) \equiv 0, \quad 1 \le \min\{i,j\} \le k-1,$$
$$(F_{d-k+1}((a^k_{pq}(x))_{k\le p,q\le d}))_{i-k,j-k} = a^{k+1}_{ij}(x), \quad k+1 \le i,j \le d,$$

and (5.28) holds. By Lemma 5.1, we see that

$$\lambda_{\min}(\tilde{a}(x)) \le a^k_{kk}(x) \le \lambda_{\max}(\tilde{a}(x)).$$

Indeed, for $k = 2$ we have

$$\begin{aligned}\lambda_{\min}(\tilde{a}(x))\langle\xi,\xi\rangle &\le \lambda_{\min}(F_d(\tilde{a}(x)))\langle\xi,\xi\rangle \le \langle\xi, F_d(\tilde{a}(x))\xi\rangle \\ &\le \lambda_{\max}(F_d(\tilde{a}(x)))\langle\xi,\xi\rangle \le \lambda_{\max}(\tilde{a}(x))\langle\xi,\xi\rangle,\end{aligned}$$

so take $\xi^t = (0,1,0,\cdots,0)$. ($k = 1$ is easier; simply take $\tilde{a}(x)$ instead of $F_d(\tilde{a}(x))$ and put $\xi^t = (1,0,\cdots,0)$.) For $k \ge 3$, we can obtain the estimate similarly by applying Lemma 5.1 iteratively. Therefore

$$\frac{1}{\lambda_{\max}(\tilde{a}(x))} \le \nu_k(x) \le \frac{1}{\lambda_{\min}(\tilde{a}(x))},\ x \in \mathbb{R}^d. \tag{5.32}$$

Next, let us introduce the set of strongly uniformly elliptic bounded symmetric matrices:

$$\mathscr{M}^{(3)}_d(\varepsilon, M_1) = \{a = \tilde{a} : a_{ii} - \sum_{j:j\neq i} |a_{ij}| \ge \varepsilon,\ i = 1,...,d,\ \|a\| \le M_1\},$$

and for $N \in \mathbb{N}$ the set of "almost" antisymmetric bounded matrices:

$$\mathscr{M}^{(1)}_d(N, M_2) = \{a : a_{ij} = -a_{ji},\ 1 \le i < j \le d,\ a_{ii} = \frac{1}{2N}\sum_{j:j\neq i}|a_{ij}|, \|a\| \le M_2\}.$$

The appearance of the diagonal term will be explained below.

Next for given $L \in \mathbb{N}$, let

$$\begin{aligned}\mathscr{M}^{(2)}_d(L, M_1) = \{a = \tilde{a} : a_{ij} = \sum_{k=1}^d \nu_k V^k_i \cdot V^k_j,\ i,j \in \{1,...,d\}, \\ \text{for } \nu_k \ge 0, V^k \in [-L,L]^d \cap \mathbb{Z}^d, k = 1,...,d,\ \|a\| \le M_1\},\end{aligned}$$

be the set of symmetric matrices with diagonalized form with coefficients $\nu_k \geq 0$ and vectors V^k integer valued.

Lemma 5.2. *Let $a \in \mathscr{M}_d(\varepsilon, M_1, M_2)$ and choose $L > [9\sqrt{d}d^2 M_1/\varepsilon]$ and $N > [3M_2/(2\varepsilon)]$, then we can find*

$$b^{(1)} \in \mathscr{M}_d^{(1)}(N, M_2+\varepsilon/3),\ b^{(2)} \in \mathscr{M}_d^{(2)}(L, M_1+\varepsilon/3),\ b^{(3)} \in \mathscr{M}_d^{(3)}(\varepsilon/3, M_1 \vee (5\varepsilon/3))$$

such that

$$a = b^{(1)} + b^{(2)} + b^{(3)}. \tag{5.33}$$

PROOF. Set $b = a - \varepsilon I$, and write $b = \tilde{b} + \hat{b}$ where $\tilde{b}$ is the symmetric part and $\hat{b}$ the antisymmetric part of b. Set

$$b_{ij}^{(1)} = -b_{ji}^{(1)} = \hat{b}_{ij},\ i \neq j, \quad b_{ii}^{(1)} = \frac{1}{2N} \sum_{j: j \neq i} |\hat{b}_{ij}|.$$

Noting that

$$b_{ii}^{(1)} \leq \frac{M_2}{2N} \leq \varepsilon/3,$$

we have $b^{(1)} \in \mathscr{M}_d^{(1)}(N, M_2 + \varepsilon/3)$. Next, if $\tilde{a} \in \mathscr{M}_d^{(3)}(\varepsilon, M_1)$ then we simply set

$$b^{(2)} = 0, \quad b^{(3)} = a - b^{(1)} \in \mathscr{M}_d^{(3)}(\varepsilon/3, M_1).$$

Otherwise, let $U^1, ..., U^d \in \mathbb{R}^d$ and $\nu^1, ..., \nu^d \in \mathbb{R}_+$ be a normalized diagonalized form of the symmetric matrix $\tilde{b}$ discussed above that satisfies (5.28), (5.29) and (5.30):

$$\tilde{b}_{ij} = \sum_{k=1}^{d} \nu_k U_i^k \cdot U_j^k,$$

with

$$|U^k| = 1, \quad 0 \leq \nu_k \leq \lambda_{\max}(\tilde{b}) \leq \lambda_{\max}(\tilde{a}) \leq \sqrt{d}\|\tilde{a}\|, \quad k = 1, ..., d.$$

Let $\bar{V}_i^k = [LU_i^k]/L \in (L^{-1}\mathbb{Z}^d) \cap [-1,1]^d$, we then have

$$|U_j^k - \bar{V}_j^k| \leq \frac{1}{L}, \quad |\bar{V}_j^k| \leq 1.$$

Define

$$b_{ij}^{(2)} = \sum_{k=1}^{d} \nu_k \bar{V}_i^k \cdot \bar{V}_j^k = \sum_{k=1}^{d} (L^{-2}\nu_k)(L\bar{V}_i^k) \cdot (L\bar{V}_j^k),$$

and

$$b^{(3)} = a - b^{(1)} - b^{(2)} = \varepsilon I + \tilde{b} - b^{(2)} + \hat{b} - b^{(1)}.$$

Note that $b^{(3)}$ is symmetric by construction with

$$b_{ij}^{(3)} = \tilde{b}_{ij} - b_{ij}^{(2)}, \quad 1 \leq i < j \leq d, \quad b_{ii}^{(3)} = \varepsilon + \tilde{b}_{ii} - b_{ii}^{(2)} - b_{ii}^{(1)},\ i = 1, ..., d.$$

By the triangle inequality we have

$$\|b^{(2)} - \tilde{b}\| \le \sum_{k=1}^{d} \max_i \sum_{j=1}^{d} \nu_k \, |U_i^k \cdot U_j^k - \bar{V}_i^k \bar{V}_j^k|,$$

where

$$|U_i^k \cdot U_j^k - \bar{V}_i^k \bar{V}_j^k| \le |\bar{V}_i^k||U_j^k - \bar{V}_j^k| + |\bar{V}_j^k||U_i^k - \bar{V}_i^k| + |U_i^k - \bar{V}_i^k||U_j^k - \bar{V}_j^k| \le \frac{2+\frac{1}{L}}{L} \le \frac{3}{L}.$$

Thus

$$\|b^{(2)} - \tilde{b}\| \le \frac{3d}{L} \sum_{k=1}^{d} \nu_k \le \frac{3d^2 \lambda_{\max}(\tilde{b})}{L} \le \frac{3d^2\sqrt{d}M_1}{L} \le \frac{\varepsilon}{3}$$

if

$$L > \frac{9\sqrt{d}d^2 M_1}{\varepsilon}.$$

This implies that

$$b_{ii}^{(3)} - \sum_{j:j\neq i} |b_{ij}^{(3)}| \ge \varepsilon - b_{ii}^{(1)} - \sum_j |b_{ij}^{(2)} - \tilde{b}_{ij}| \ge \varepsilon/3.$$

Noting that $\|b^{(3)}\| \le \varepsilon + \|\tilde{b} - b^{(2)}\| + \|\hat{b} - b^{(1)}\| \le 5\varepsilon/3$, the above implies

$$b^{(3)} \in \mathscr{M}_d^{(3)}(\varepsilon/3, 5\varepsilon/3), \quad b^{(2)} \in \mathscr{M}_d^{(2)}(L, M_1 + \varepsilon/3),$$

and ends the proof. □

In view of the additive rule, it thus suffices to find a collection of cycles Γ_n^k such that $F_{ij}^{k,n}(\,\cdot\,)$ converges locally in $L^1(\mathbb{R}^d)$ to $b_{ij}^{(k)}$, for each $k = 1,2,3$.

Examples 1 and 2 of the previous section imply the following.

Lemma 5.3. *Let M, N and L be fixed and $b^n : \mathscr{S}_n \to [0,M]$ be such that*

$$\lim_{n\to\infty} \|b^n(\,\cdot + y/n) - b^n\|_K = 0, \quad \forall y \in \mathbb{Z}^d,\ \forall K \subset \mathbb{R}^d \text{ compact}. \tag{5.34}$$

a) Referring to Example 1, for given fixed $V \in [-L,L]^d \cap \mathbb{Z}^d$, take

$$\Gamma_n = \{\gamma_x^n = (x, x+V/n, x), x \in \mathscr{S}_n\}, \text{ with weights } \alpha_n(\gamma_x^n) = b^n(x), x \in \mathscr{S}_n,$$

and set $a^n \in \mathscr{M}_d^{(3)}(L,M)$ by

$$a_{ij}^n(x) = b^n([x]_n) V_i \cdot V_j,\ 1 \le i,j \le d,\ x \in \mathbb{R}^d.$$

Then for every $K \subset \mathbb{R}^d$ compact, $\lim_{n\to 0} \|F_{ij}^n - a_{ij}^n\|_K = 0$.
b) Referring to Example 2, for fixed N and $l \neq m \in \{1,\dots,d\}$, take

$$\Gamma_n = \{\gamma_{N,x}^{n,(l,m)}, x \in \mathscr{S}_n\}, \text{ with weights } \alpha_n(\gamma_{N,x}^{n,(l,m)}) = \frac{b^n(x)}{4N^2},\ x \in \mathscr{S}_n,$$

and set $a^n \in \mathscr{M}_d^{(1)}(N,M)$ by

$$a_{l,l}^n(x) = a_{m,m}^n(x) = \frac{b^n([x]_n)}{2N}, \quad a_{l,m}^n(x) = -a_{m,l}^n(x) = b^n([x]_n), \quad a_{ij}^n(x) = 0, i,j \notin \{l,m\},$$

for $x \in \mathbb{R}^d$. *Then for every* $K \subset \mathbb{R}^d$ *compact,* $\lim_{n\to 0} \|F^n_{ij} - a^n_{ij}\|_K = 0$.

PROOF. Since V is constant, we first see that $R^{2,n}_{ij}(z) = R^{3,n}_{ij}(z) = 0$, cf. (5.17). Next using the fact that $|b^n(x)| \le M$, $|V_i| \le L$ we get in view of (5.11) and (5.21) writing $\|y\|_1 = \sum_{i=1}^d |y_i|$,

$$|R^{1,n}_{ij}(z)| \le L^2 \max_{y:\|y\|_1\le L} |b^n(z+y/n) - b^n(z)| \le L^2 \sum_{y:\|y\|_1\le L} |b^n(z+y/n) - b^n(z)|,$$

and

$$|R^{4,n}_{ij}(z)| \le 4N^2 \max_{y:\|y\|_1\le 4N} |b^n(z+y/n) - b^n(z)| \le 4N^2 \sum_{y:\|y\|_1\le 4N} |b^n(z+y/n) - b^n(z)|.$$

Using our assumption this yields

$$\lim_{n\to 0} \|R^{k,n}_{ij}\|_K = 0, \quad k = 1,4,$$

and implies the result. □

Next we want to extend our result to piecewise constant vectors V. Recall the definition of $r_n = [n^{1-\beta}]/n \in (1/n)\mathbb{Z}_+$ for some $\beta \in (0,1)$.

Lemma 5.4. *a) Let* $b^n : \mathscr{S}_n \to [-M,M]$ *be* r_n*-piecewise constant. Then for each compact* $K \subset \mathbb{R}^d$ *and fixed* $y \in \mathbb{Z}^d$*,*

$$\|b^n(\cdot) - b^n(\cdot + y/n)\|_K \le \frac{\|y\|_1 C_K M}{[n^{1-\beta}]} \tag{5.35}$$

for some $C_K < \infty$ *depending on the diameter of K, where n is taken large enough so that* $[n^{1-\beta}] \ge 2\|y\|_1$.
b) Referring to Example 1, let $b^n : \mathscr{S}_n \to [0,M]$ *and* $V^n : \mathscr{S}_n \to [-L,L]^d \cap \mathbb{Z}^d$ *be* r_n*-piecewise constant and take*

$$\Gamma_n = \{\gamma^n_x = (x, x+V^n(x)/n, x), x \in \mathscr{S}_n\}, \text{ with weights } \alpha_n(\gamma^n_x) = b^n(x), x \in \mathscr{S}_n,$$

and set $a^n \in \mathscr{M}^{(2)}_d(L,M)$ *by*

$$a^n_{ij}(x) = b^n([x]_n) V^n_i([x]_n) \cdot V^n_j([x]_n),\ 1 \le i,j \le d,\ x \in \mathbb{R}^d.$$

Then for every $K \subset \mathbb{R}^d$ *compact,* $\lim_{n\to 0} \|F^n_{ij} - a^n_{ij}\|_K = 0$.

PROOF. a) Simply note that

$$b^n(z) = \sum_{x\in J_{r_n}} b^n(x) 1_{Q(x,r_n)}(z)$$

and therefore

$$\|b^n(\cdot) - b^n(\cdot + y/n)\|_K \le M \sum_{x\in J_{r_n}} \Big| \int_K (1_{Q(x,r_n)}([z]_n) - 1_{Q(x,r_n)}([z]_n + y/n))\, dz \Big|.$$

Now for the interior points of $Q(x,r_n)$ such that

$$IQ(x,r_n,\|y\|_1/n) := \{z \in Q(x,r_n) : \|y\|_1/n \le \min_i (z_i - x_i) \le \max_i (z_i - x_i) \le r_n - \|y\|_1/n\},$$

we clearly have

$$1_{Q(x,r_n)}(z) - 1_{Q(x,r_n)}(z+y/n) = 0, \quad z \in IQ(x,r_n,\|y\|_1/n).$$

So all what remains are the boundary terms

$$BQ(x,r_n,\|y\|_1/n) := Q(x,r_n) \setminus IQ(x,r_n,\|y\|_1/n)$$

with

$$\sum_{x\in J_r} \Big| \int_K (1_{BQ(x,r_n,\|y\|_1/n)}([z]_n)\,dz \Big| \le \frac{\|y\|_1 C_K}{[n^{1-\beta}]}.$$

b) In view of (5.7), (5.11)–(5.12) we have

$$|R^{1,n}_{ij}(z)| \le CL^{d+1} \max_{y:\|y\|_1\le L} |b^n(z+y/n) - b^n(z)| \le CL^{d+1} \sum_{y:\|y\|_1\le L} |b^n(z+y/n) - b^n(z)|,$$

$$|R^{2,n}_{ij}(z)| \le CML^{d} \max_{y:\|y\|_1\le L} |V^n_j(z+y/n) - V^n_j(z)| \le CML^{d} \sum_{y:\|y\|_1\le L} |V^n_j(z+y/n) - V^n_j(z)|,$$

and a) shows that $\|R^{1,n}_{ij}\|_K$ and $\|R^{2,n}_{ij}\|_K \to 0$ as $n \to \infty$. Next note that (5.17) implies $R^{3,n}_{ij}(z) = 0$ if $z \in \cup_{x\in J_{r_n}} IQ(x,r_n,L/n)$ and by (5.7), (5.13), $|R^{3,n}_{ij}(z)| \le CML^{d+1}$ if $z \in \cup_{x\in J_{r_n}} BQ(x,r_n,L/n)$. As in a) this shows $\|R^{3,n}_{ij}\|_K \to 0$ as $n \to \infty$. □

With these preparations we can easily construct our approximation. Consider the r_n-piecewise approximation a_n of the matrix a, cf. (5.24) and (5.26), set

$$b^n(x) = a^n(x) - \varepsilon I, \quad x \in \mathscr{S}_n.$$

Denote by $\tilde b^n(x)$ and $\hat b^n(x)$ the symmetric and antisymmetric part of $b^n(x)$. Choose $L = [9\sqrt{d}d^2 M_1/\varepsilon] + 1$, $N = [3M_2/(2\varepsilon)] + 1$, and define $b^{(i),n}(x)$ as in Lemma 5.2, that is

$$a^n(x) = b^{(1),n}(x) + b^{(2),n}(x) + b^{(3),n}(x)$$

where $b^{(1),n}(x) \in \mathscr{M}^{(1)}_d(N,M), b^{(3),n}(x) \in \mathscr{M}^{(3)}_d(\varepsilon/3,M)$, and

$$b^{(2),n}_{ij}(x) = \sum_{k=1}^{d} \lambda^n_k(x) V^{k,n}_i(x) \cdot V^{k,n}_j(x) \in \mathscr{M}^{(2)}_d(L,M).$$

Here we set $M := (M_1+\varepsilon/3) \vee (M_2+\varepsilon/3) \vee 5\varepsilon/3$.

Note that by construction, $b^{(1),n}, b^{(3),n}, \lambda^n_k$ and $V^{k,n}$ are bounded and r_n-piecewise constant, so by Lemma 5.4 a), for all compact $K \subset \mathbb{R}^d$ and fixed $y \in \mathbb{Z}^d$,

(5.36)
$$\|b^{(1),n}_{ij}(\,\cdot+y/n) - b^{(1),n}_{ij}\|_K \le \frac{\|y\|_1 C_K M}{[n^{1-\beta}]}, \quad \|b^{(3),n}_{ij}(\,\cdot+y/n) - b^{(3),n}_{ij}\|_K \le \frac{\|y\|_1 C_K M}{[n^{1-\beta}]},$$

(5.37)
$$\|V^{k,n}_j(\,\cdot+y/n) - V^{k,n}_j\|_K \le \frac{\|y\|_1 C_K L}{[n^{1-\beta}]},$$

(5.38)
$$\|\lambda^n_k(\,\cdot+y/n) - \lambda^n_k\|_K \le \frac{\|y\|_1 C_K M}{[n^{1-\beta}]}.$$

For $b^{(1),n}(x) \in \mathscr{M}_d^{(1)}(N,M)$ consider $(\Gamma^{(1),n}, \alpha^n)$ as follows

$$\Gamma^{(1),n} = \{\gamma_{N,x}^{n,(i,j)},\ \gamma_{N,x}^{n,(j,i)},\ 1 \le i < j \le d, x \in \mathscr{S}_n\}$$

with weights

$$\alpha^n(\gamma_{N,x}^{n,(i,j)}) = \frac{(b_{ij}^{(1),n}(x))^+}{4N^2} = \frac{(\hat{a}_{ij}^n(x))^+}{4N^2}, \quad \alpha^n(\gamma_{N,x}^{n,(j,i)}) = \frac{(b_{ij}^{(1),n}(x))^-}{4N^2} = \frac{(\hat{a}_{ij}^n(x))^-}{4N^2},$$

where $a^+ := \max\{a,0\}$ and $a^- = \max\{(-a),0\}$ for $a \in \mathbb{R}$. Then, in view of (5.36), the additive rule and Lemma 5.3 b), we see that the corresponding $F_{ij}^{(1),n}$ satisfies

$$\lim_{n\to\infty} \|F_{ij}^{(1),n} - b_{ij}^{(1),n}\|_K = 0.$$

Next consider $(\Gamma^{(2),n}, \alpha^n)$ of the form

$$\Gamma^{(2),n} = \{\gamma_x^{k,n} = (x, x+V^{k,n}(x)/n, x),\ \text{ with weights }\ \alpha^n(\gamma_x^{k,n}) = \lambda_k^n(x),\ k = 1,...,d,\ x \in \mathscr{S}_n\},$$

then in view of (5.37) and (5.38), Lemma 5.4 b) and the additive rule, we see that the corresponding $F_{ij}^{(2),n}$ satisfies

$$\lim_{n\to\infty} \|F_{ij}^{(2),n} - b_{ij}^{(2),n}\|_K = 0.$$

Finally for

$$b^{(3),n}(x) = a^n(x) - b^{(1),n}(x) - b^{(2),n}(x) \in \mathscr{M}_d^{(3)}(\varepsilon/3, M),$$

take $(\Gamma^{(3),n}, \alpha^n)$ of the form

$$\Gamma^{(3),n} = \{\gamma_{ij}^{n,\pm}(x) = (x, x+e_i/n \pm e_j/n, x), 1 \le i < j \le d, \gamma_i^n(x) = (x, x+e_i/n, x), x \in \mathscr{S}_n\}$$

with weights

$$\alpha^n(\gamma_{ij}^{n,+}(x)) = (b_{ij}^{(3),n}(x))^+, \quad \alpha^n(\gamma_{ij}^{n,-}(x)) = (b_{ij}^{(3),n}(x))^-,$$

$$\alpha^n(\gamma_i^n(x)) = b_{ii}^{(3),n}(x) - \sum_{j:j\neq i} |b_{ij}^{(3),n}(x)| \ge \varepsilon/3, \quad x \in \mathscr{S}_n.$$

We call $\gamma_i^n(x)$ a *nearest neighbor cycle* and $\gamma_{ij}^{n,\pm}(x)$ a *diagonal cycle*.

Then using (5.36), the additive rule and the Lemma 5.3 a), we see that the corresponding $F_{ij}^{(3),n}$ satisfies

$$\lim_{n\to\infty} \|F_{ij}^{(3),n} - b_{ij}^{(3),n}\|_K = 0.$$

Putting things together we have the following.

Theorem 5.5. *For any measurable map $a : \mathbb{R}^d \to \mathscr{M}_d(\varepsilon, M_1, M_2)$, we can find a sequence (Γ_n, α_n) that satisfies* (2.2) *in Assumption 2.1 and* (2.8), (2.9) *in Assumption 2.3, such that the corresponding $F_{ij}^n(x)$ converges to $a_{ij}(x)$ locally in $L^1(\mathbb{R}^d)$.*

Furthermore, writing $\Gamma_n = \{\gamma_{n,i}, i \in I\}$*, each cycle* $\gamma_{i,n}$ *is either a two cycle or a rotational cycle that satisfies*

$$\alpha_n(\gamma_{n,i}) \le \max(M_1, M_2),\ \ell(\gamma_{n,i}) \le \max(2, 8([3M_2/(2\varepsilon)]+1)),$$
$$Range(\gamma_{n,i}) \le \max(2, [9\sqrt{d}d^2 M_1/\varepsilon]+1)/n, \quad \forall i, n,$$

and (2.8) *is satisfied with* $N = 1$ *and* $\delta = \varepsilon/3$.

Note that $\alpha_n(\gamma_{i,n}) \ge 0$ in the above construction. However, by neglecting cycles with $\alpha_n(\gamma_{i,n}) = 0$, we may say that weights of cycles in Γ_n are all positive.

Remark 5.6. (i) Our construction is very explicit. For example, when approximating a symmetric diffusion matrix in [SZ], they have additional procedure of smoothing the matrix by convolution, whereas we can avoid this procedure. We think that our construction is practical in that it is useful when simulating diffusions in divergence form.
(ii) As we have seen, once the lattice approximation of the symmetric part is computed, the antisymmetric part can be easily dealt with rotational cycles which are just translates of a fixed cycle. In case of a strongly uniformly elliptic bounded symmetric matrix $\tilde{a}(x)$, we do not need to bring the matrix in diagonalized form and we can restrict ourselves to nearest neighbor and diagonal cycles. Otherwise, in view of the Feshbach map, we can avoid the computation of eigenvalues and eigenvectors.

(iii) Although we do not investigate the convergence speed of our approximation it is very natural to take $\beta = 1/2$, since for "nice" $a(x)$ we expect

$$\|a^n_{ij} - a_{ij}\|_K = O(n^{-\beta}),$$

whereas

$$\|a^n_{ij} - a^n_{ij}(\cdot + y/n)\|_K = O(n^{-1+\beta}).$$

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Bibliography

[Al] Alexopoulos, G.K. Random walks on discrete groups of polynomial volume growth. *Ann. Probab.* **30** (2002), 723–801.

[BFS] Bach, V.; Fröhlich J; Sigal M. Renormalization group analysis of spectral problems in Quantum Field Theory. *Adv. in Math.* **137** (1998) 205-298.

[BBCK] Barlow, M.T.; Bass, R.F.; Chen, Z.-Q.; Kassmann, M. Non-local Dirichlet forms and symmetric jump processes. *Trans. Amer. Math. Soc.* **361** (2009), 1963–1999.

[BaH] Barlow, M.T.; Hambly, B.M. Parabolic Harnack inequality and local limit theorem for percolation clusters. *Elec. J. Probab.* **14**, (2009) 1–26.

[BGK] Barlow, M.T.; Kumagai, T.; Grigor'yan, A. On the equivalence of parabolic Harnack inequalities and heat kernel estimates. *J. Math. Soc. Japan*, to appear. http://www.kurims.kyoto-u.ac.jp/∼kumagai/bgkhar-resubmit.pdf

[BK] Bass, R.F.; Kumagai, T. Symmetric Markov chains on $\mathbb{Z}^d$ with unbounded range. *Trans. Amer. Math. Soc.* **360** (2008), 2041–2075.

[BKU] Bass, R.F.; Kumagai, T.; Uemura, T. Convergence of symmetric Markov chains on $\mathbb{Z}^d$. *Probab. Theory Relat. Fields* **148** (2010), 107–140.

[Birk] Birkhoff, G. Three observations on linear algebra, *Univ. Nac. Tucumán. Revista A.* **5** (1946), 147–151.

[CKS] Carlen, E.A.; Kusuoka, S.; Stroock, D.W. Upper bounds for symmetric Markov transition functions. *Ann. Inst. Henri Poincaré-Probab. Statist.* **23** (1987), 245-287.

[DK] Deuschel, J-D.; Kösters, H. The quenched invariance principle for random walks in random environments admitting a bounded cycle representation. *Ann. Inst. Henri Poincaré* **44** (2008), 574–591.

[Du] Dungey, N. Properties of centered random walks on locally compact groups and Lie groups. *Rev. Mat. Iberoamericana* **23** (2007), 587–634.

[FS] Fabes, E.B.; Stroock, D.W. A new proof of Moser's parabolic Harnack inequality using the old ideas of Nash. *Arch. Mech. Rat. Anal.* **96** (1986), 327–338.

[FOT] Fukushima, M.; Oshima, Y.; Takeda, M. *Dirichlet Forms and Symmetric Markov Processes.* deGruyter, Berlin, 1994.

[MR] Ma, Z.-M.; Röckner, M. *Introduction to the Theory of (Non-Symmetric) Dirichlet Forms.* Springer-Verlag, New York, 1992.

[Ma] Mathieu, P. Carne-Valopoulos bounds for centered random walks. *Ann. Probab.* **34** (2006), 987–1011.

[SC] Saloff-Coste, L. *Aspects of Sobolev-Type inequalities.* Lond. Math. Soc. Lect. Notes 289, Cambridge Univ. Press, 2002.

[SV] Stroock, D.W.; Varadhan, S.R.S. *Multidimensional Diffusion Processes.* Springer-Verlag, Berlin, 1979.

[SZ] Stroock, D.W.; Zheng, W. Markov chain approximations to symmetric diffusions. *Ann. Inst. Henri. Poincaré-Probab. Statist.* **33** (1997), 619-649.

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