5 Strongly recurrent case

5.1 Framework and the main theorem

$(X, d, \mu, \mathcal{E})$: MMD space or the weighted graph

It is called a *resistance form* if $\mathcal{F} \subset C(X)$ and

$$\sup\{\frac{|u(p)-u(q)|^2}{\mathcal{E}(u,u)} : u \in \mathcal{F}, \mathcal{E}(u,u) > 0\} < \infty, \qquad \forall p, q \in X. \tag{5.1}$$

Define $R(p,q) =$ (LHS of (5.1)) if $p \neq q$ and $R(p,p) = 0$.

R is a metric, called a *resistance metric*. By (5.1), the following key inequality holds.

$$|f(x) - f(y)|^2 \leq R(x,y)\mathcal{E}(f,f), \qquad \forall f \in \mathcal{F}. \tag{5.2}$$

The next lemma shows that $R(p,q)$ is the effective resistance between p and q.

Lemma 5.1

$$R(p,q) = (\inf\{\mathcal{E}(f,f) : f(p) = 1, f(q) = 0, f \in \mathcal{F}\})^{-1}. \tag{5.3}$$

PROOF. We can take $f(x) = 1, f(y) = 0$ by linear transform if u is not const. So,

$$\begin{aligned} R(x,y) &= \sup\{\frac{|u(x)-u(y)|^2}{\mathcal{E}(u,u)} : u \in \mathcal{F}, \mathcal{E}(u,u) > 0\} \\ &= \sup\{\frac{1}{\mathcal{E}(f,f)} : f \in \mathcal{F}, f(x) = 1, f(y) = 0\} \\ &= (\inf\{\mathcal{E}(f,f) : f(x) = 1, f(y) = 0, f \in \mathcal{F}\})^{-1}. \quad \square \end{aligned}$$

Examples. The following are resistance forms.

- Weighted graphs
- For the Dirichlet form on $\mathbb{R}^1$ that corresponds to Brownian motion.
- Dirichlet forms on the Sierpinski gasket, nested fractals

  (and 'typical' finitely ramified fractals).
- Dirichlet forms on the 2-dimensional Sierpinski carpet.

(I) Volume growth condition ($VG(\Psi_-)$): $\exists \alpha < \beta \vee \bar{\beta}, C > 0$ s.t.

$$V(x,r) \le C \left(\frac{r}{s}\right)^{\alpha} V(x,s) \qquad \forall x \in X,\ \forall r \ge s > 0. \tag{$VG(\Psi_-)$}$$

(II) Resistance upper and lower bound of order Ψ $(RU(\Psi)), (RL(\Psi))$:

$\exists C_1, C_2 > 0$ s.t. $\forall x, y \in X$,

$$R(x,y) \le C_1 \frac{\Psi(d(x,y))}{\mu(B(x,d(x,y)))}, \tag{$RU(\Psi)$}$$

$$C_1 \frac{\Psi(d(x,y))}{\mu(B(x,d(x,y)))} \le R(x,y). \tag{$RL(\Psi)$}$$

Theorem 5.2 *Let $(X, d, \mu, \mathcal{E})$ be a resistance form on a MMD space or a weighted graph. Assume $(VG(\Psi_-))$. Then,*

$$(HK(\Psi)) \Leftrightarrow (RU(\Psi)) + (RL(\Psi)) \Leftrightarrow (RL(\Psi)) + (PI(\Psi)). \tag{5.4}$$

When (5.4) holds, it is strongly recurrent in the sense that $\exists p_1 > 0$ s.t.

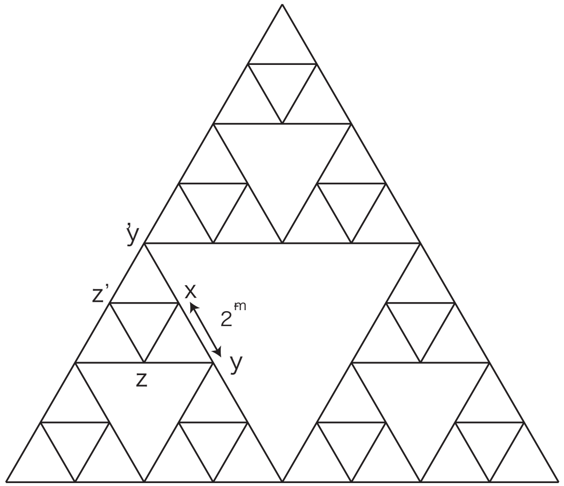
$$P^x(\sigma_y < \tau_{B(x,2r)}) \geq p_1, \qquad \forall x \in X, r > 0,\ y \in B(x,r), \tag{5.5}$$

where $\sigma_A = \inf\{t \geq 0 : X_t \in A\}$ and $\tau_A = \inf\{t \geq 0 : X_t \notin A\}$.

When X is a tree, we have a simpler equivalence condition as follows.

Corollary 5.3 *Let (X, μ) be a weighted graph with $c_1 \leq \mu_{xy} \leq c_2$ for all $x \sim y$. Assume that X is a tree. Then,*

$$(VG(\beta_-)) + (HK(\beta)) \Leftrightarrow [V(x, d(x,y)) \asymp d(x,y)^{\beta-1} \quad \forall x, y].$$



Check $(RU(\beta)) + (RL(\beta))$ for the Sierpinski gasket F

$\beta = \log 5/\log 2$, $\quad F_n$: set of vertices of triangles of side 2^{-n}

$$\mathcal{E}(f,f) = c \lim_{n\to\infty} (5/3)^n \sum_{a\sim b\in F_n} (f(a)-f(b))^2, \qquad f \in \Lambda^{\beta/2}_{2,\infty}(F).$$

$x, y, z, y', z' \in F_m$ as in the figure, h_m: $h_m(x) = 1, h_m(y) = h_m(z) = h_m(y') = h_m(z') = 0$

and harmonic outside. $\Rightarrow \mathcal{E}(h_m, h_m) = c'(5/3)^m$.

Then, $\mathcal{E}(h_m) \geq R(x,y)^{-1} =: \mathcal{E}(f) \geq \mathcal{E}_\Delta(f) \geq c\mathcal{E}_\Delta(h_m) = c\mathcal{E}(h_m)/2$.

So $R(x,y) \asymp (5/3)^{-m} = 2^{-m(\beta - \log 3/\log 2)}$. $\square$

5.2 Proof of Theorem 5.2: $(RU(\Psi)) + (RL(\Psi)) \Rightarrow (HK(\Psi))$

The flowchart of the proof is similar to that of Proposition 4.1.

$$\text{By } (VG(\Psi_-)), \quad \exists c > 0 \text{ s.t.} \qquad \frac{\Psi(s)}{V(x,s)} \leq c\frac{\Psi(r)}{V(x,r)} \qquad \forall r > s > 0. \tag{5.6}$$

Indeed, by $(VG(\Psi_-))$, we have

$$\frac{V(x,r)}{V(x,s)} \leq c(\frac{r}{s})^{\alpha} < c(\frac{r}{s})^{\beta\wedge\bar{\beta}} \leq c\frac{\Psi(r)}{\Psi(s)}, \qquad \forall r > s > 0, \quad \text{which implies (5.6).}$$

We now give the proof of $(RU(\Psi)) + (RL(\Psi)) \Rightarrow (HK(\Psi))$ step by step.

STEP A: PROOF OF $(RU(\Psi)) \Rightarrow (DUHK(\Psi))$. Let $f_t(y) = p_t(x,y)$ and

$$\varphi(t) := ||f_t||_2^2 = p_{2t}(x,x) = f_{2t}(x). \tag{5.7}$$

Since $\int_{B(x,r)} f_t d\mu \leq 1$ for $r > 0$, $\exists y = y(t,r) \in B(x,r)$ with $f_t(y) \leq V(x,r)^{-1}$.

Using (5.2),

$$\frac{1}{2}f_t(x)^2 \leq f_t(y)^2 + |f_t(x) - f_t(y)|^2 \leq \frac{1}{V(x,r)^2} + \mathcal{E}(f_t,f_t)R(x,y).$$

Since $R(x,y) < c_1\Psi(r)/V(x,r)$, which is due to $(RU(\Psi))$, it follows that

$$\frac{c_1\Psi(r)}{V(x,r)}\mathcal{E}(f_t,f_t) \geq \frac{1}{2}\varphi(t/2)^2 - \frac{1}{V(x,r)^2}.$$

Hence

$$\varphi'(t) = -2\mathcal{E}(f_t,f_t) \leq \frac{2V(x,r)^{-1} - \varphi(t/2)^2 V(x,r)}{c_1\Psi(r)}. \tag{5.8}$$

Noting that $-\varphi(t/2)^2 \leq -\varphi(t)^2$, which is due to the fact $\varphi'(t) = -2\mathcal{E}(f_t,f_t) \leq 0$, we integrate (5.8) over $[t,2t]$. Then,

$$-\varphi(t) \leq \varphi(2t) - \varphi(t) \leq \frac{2t}{c_1\Psi(r)V(x,r)} - \frac{t\varphi(t)^2 V(x,r)}{c_1\Psi(r)}.$$

Rearranging this, we have

$$t\varphi(t)^2 V(x,r)^2 \leq 2t + c_1\Psi(r)V(x,r)\varphi(t) \leq (4t) \vee (2c_1\Psi(r)V(x,r)\varphi(t)).$$

Thus, we obtain $\varphi(t) \leq (2/V(x,r)) \vee (2c_1\Psi(r)/(tV(x,r)))$. Taking $r = \Psi^{-1}(t)$ and using the doubling properties of Ψ and V, we obtain $(DUHK(\Psi))$. $\square$

STEP B: PROOF OF $(VG(\Psi_-)) + (RU(\Psi)) + (RL(\Psi)) \Rightarrow (E(\Psi))$.

Lemma 5.4 *Assume* $(VG(\Psi_-)), (RU(\Psi))$ *and* $(RL(\Psi))$. *Then,*

$$\frac{c_1\Psi(r)}{V(x,r)} \le R(x, B(x,r)^c) \le \frac{c_2\Psi(r)}{V(x,r)} \qquad \forall r > 0,\ \forall x \in X. \tag{5.9}$$

PROOF. First, take $y, z \in B(x,r)$ with $d(y,z) = \lambda r$, $\lambda \le 1$. By (5.2) and $(RU(\Psi))$,

$$|f(y) - f(z)|^2 \le R(y,z)\mathcal{E}(f,f) \le \frac{c_2\Psi(\lambda r)\mathcal{E}(f,f)}{V(x,\lambda r)}, \qquad \forall f \in \mathcal{F}. \tag{5.10}$$

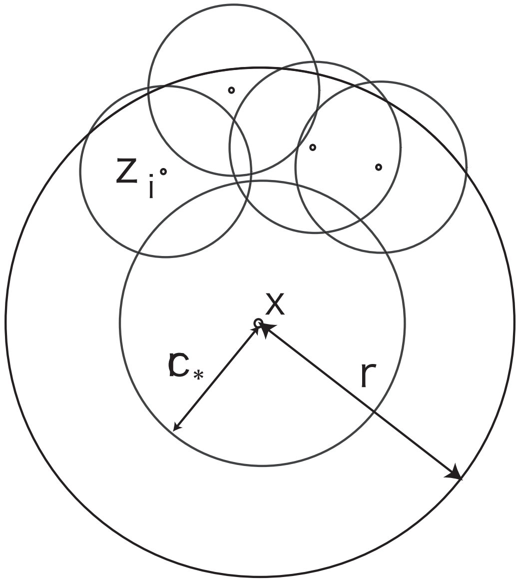
Let $z \in X$ be s.t. $c_* r \le d(x,z) \le r$ for $\exists c_* < 1$.

If h_z harm. fu. on $X \setminus \{x, z\}$ with $h_z(z) = 0$, $h_z(x) = 1$ then $\mathcal{E}(h_z, h_z) = R(x,z)^{-1}$.

Applying (5.6), (5.10) and $(RL(\Psi))$, we have, if $d(y,z) = \lambda r$,

$$|h_z(y)|^2 = |h_z(y) - h_z(z)|^2 \le \frac{c_2\Psi(\lambda r)}{V(x,\lambda r)R(x,z)} \le \frac{c_3\Psi(\lambda r)V(x,c_* r)}{V(x,\lambda r)\Psi(c_* r)}.$$

So $\exists\lambda_1$ s.t. $d(y,z) \le \lambda_1 r$ implies that $h_z(y) \le \frac{1}{2}$.



Now use (VD) to cover $B(x,r) \setminus B(x,c_* r)$ by balls $B(z_i, \lambda_1 r)$, $1 \le i \le M$, with $c_* r \le d(x,z_i) \le r$. (M dep. only on the volume doubling constant.)

Let $g := \min h_{z_i}$, $h := 2(g - \frac{1}{2})^+ \cdot 1_{B(x,r)}$. Then $h(x) = 1$, $h = 0$ on $B(x, c_* r)^c$, so that

$$R(x, B(x,r)^c)^{-1} \le \mathcal{E}(h,h) \le 4 \sum_i \mathcal{E}(h_{z_i}, h_{z_i}) \le 4M(\min_i R(x,z_i))^{-1}$$

$$\le \frac{c_4 V(x, c_* r)}{\Psi(c_* r)} \le \frac{c_5 V(x,r)}{\Psi(r)} \qquad \Rightarrow \text{1st ineq. of (5.9).}$$

The 2nd ineq. of (5.9) is clear: $(RU(\Psi))+$ $[R(x, B(x,r)^c) \le R(x,y), \forall y \in \partial B(x,r)]$. $\square$

Proof of $(E(\Psi))$. $B := B(x_0, r)$, $(\mathcal{E}_B, \mathcal{F}_B)$: part of the Dirichlet form $\mathcal{F}_B \subset \{f \in \mathcal{F} : f(x) = 0 \text{ on } x \in B^c\}$. By (5.2) and $(RU(\Psi))$, we have

$$\sup_{x\in B} |f(x)|^2 \le \frac{c_1\Psi(r)}{V(x,r)}\mathcal{E}(f,f), \qquad \forall\, f \in \mathcal{F}_B. \tag{5.11}$$

(5.11)+ the Riesz theorem $\Rightarrow \exists g_B(\cdot,\cdot)$ Green kernel s.t. $\mathcal{E}(g_B(x,\cdot), f) = f(x)$, $\forall f \in \mathcal{F}_B$.

$g_B(x,y) = g_B(y,x)$ and $g_B(x,x) > 0\ \forall x, y \in B$.

$p_x(y) := g_B(x,y)/g_B(x,x)$. Then p_x is an equilibrium potential for $R(x, B^c)$, so

$$R(x,B^c)^{-1} = \mathcal{E}(p_x, p_x) = g_B(x,x)^{-1}. \tag{5.12}$$

$$\text{Since } p_x(y) \le 1\ \ \forall y \in X, \qquad g_B(x,y) \le g_B(x,x)\ \ \forall x, y \in X. \tag{5.13}$$

On the other hand, $R(x,B^c) \le R(x,y)\ \forall y \in B^c$, so $g_B(x,x) \le c_1\Psi(r)/V(x,r)$.

Since $E^{x_0}[\tau_{B(x_0,r)}] = \int_B g_B(x_0,y)d\mu(y)$, we have, using (5.13),

$$E^{x_0}[\tau_{B(x_0,r)}] \le \frac{c_1\Psi(r)}{V(x_0,r)}V(x_0,r) \le c_1\Psi(r) \quad \Rightarrow \quad \text{2nd ineq. of } E(\Psi).$$

Next, by (5.2) and the reproducing property of g_B,

$$|g_B(x_0,x_0) - g_B(x_0,y)|^2 \le \mathcal{E}(g_B,g_B)R(x_0,y) = g_B(x_0,x_0)R(x_0,y), \qquad \forall y \in B.$$

Thus, by (5.12), $|1-p_{x_0}(y)|^2 \le \frac{R(x_0,y)}{R(x_0,B^c)}$. Now using Lemma 5.4, $\exists\delta > 0$ s.t.

$$p_{x_0}(y) = \frac{g_B(x_0,y)}{g_B(x_0,x_0)} \ge 1/2, \qquad \forall y \in B(x_0,\delta r). \tag{5.15}$$

By (5.12) and Lemma 5.4, $g_B(x_0,x_0) = R(x_0,B^c) \ge c_2\Psi(r)/V(x_0,r)$.

Combining this with (5.15), $g_B(x_0,y) \ge \frac{c_3\Psi(r)}{V(x_0,r)}$, $\forall y \in B(x_0,\delta r)$. So,

$$\mathbb{E}^{x_0}[\tau_{B(x_0,r)}] = \int_B g_B(x_0,y)d\mu(y) \ge \frac{c_3\Psi(r)}{V(x_0,r)}V(x_0,\delta r) \ge c_4\Psi(r),$$

where $c_4 > 0$ depends on δ. We thus obtain the 1st ineq. of $(E(\Psi))$. □

Remark. (5.15) implies immediately (5.5). This implies (EHI) by Lemma 1.6 in [6]. Thus, $(RU(\Psi)) + (RL(\Psi)) \Rightarrow (HK(\Psi))$ is proved by Prop 4.1 and Prop 4.3 (Step A above was not needed). But we do not choose this way.

Step C: Proof of $(VD) + (DUHK(\Psi)) + (E(\Psi)) \Rightarrow (UHK(\Psi))$.

This step is the same as Step 1 and Step 2 in the proof of Prop 4.1.

Step D: Proof of $(VD) + (ELD(\Psi)) \Rightarrow (DLHK(\Psi))$.

This step is the same as Step 3 in the proof of Prop 4.1.

Step E: Proof of $(VG(\Psi_-)) + (RU(\Psi)) + (DLHK(\Psi)) \Rightarrow (NLHK(\Psi))$.

First, $(RU(\Psi)) \Rightarrow (DUHK(\Psi))$ as in Step A. Since $p_t(x,x) = \|p_{t/2}(\cdot,x)\|_2^2$, we have

$\partial_t p_t(x,x) = 2(\Delta p_{t/2}(\cdot,x), p_{t/2}(\cdot,x)) = -2\mathcal{E}(p_{t/2}(\cdot,x), p_{t/2}(\cdot,x))$.

Thus, using (5.2) and Prop 9.9 (time derivative), we have

$$|p_t(x,y) - p_t(x,y')|^2 \le R(y,y')\mathcal{E}(p_t(\cdot,x), p_t(\cdot,x)) \le \frac{\Psi(d(y,y'))}{V(y,d(y,y'))} \cdot \frac{c_1}{tV(x,\Psi^{-1}(t))}.$$

Using this and $(DLHK(\Psi))$,

$$\begin{aligned} p_t(x,y) &\geq p_t(x,x) - |p_t(x,x) - p_t(x,y)| \\ &\geq \frac{c_2}{V(x,\Psi^{-1}(t))} - \left\{ \frac{\Psi(d(x,y))}{V(x,d(x,y))} \cdot \frac{c_1}{tV(x,\Psi^{-1}(t))} \right\}^{1/2} \\ &= \frac{c_2}{V(x,\Psi^{-1}(t))^{1/2}} \left(\frac{1}{V(x,\Psi^{-1}(t))^{1/2}} - c_3 \left(\frac{\Psi(d(x,y))}{tV(x,d(x,y))} \right)^{1/2} \right). \end{aligned}$$

Taking c_4 large, we have $\frac{1}{2V(x,\Psi^{-1}(t))^{1/2}} \geq c_3 \left(\frac{\Psi(d(x,y))}{tV(x,d(x,y))} \right)^{1/2}$ if $\Psi(d(x,y)) \leq c_4 t$ holds. Here we used (5.6). We thus obtain the result. □

Step F: Proof of $(NLHK(\Psi)) \Rightarrow (LHK(\Psi))$.

This step is the same as Step 5 in the proof of Prop 4.1.

Combining Step A–F, the proof of $(RU(\Psi)) + (RL(\Psi)) \Rightarrow (HK(\Psi))$ is completed.

8.2 Equivalence to $(UHK(\beta))$

In [36], A. Grigor'yan proved various equivalence conditions for $(UHK(\beta))$ under (VD).

- Faber-Krahn ineq $(FK(\beta))$: $\exists \nu > 0$ s.t. $\forall B_r \subset X$ and $\forall$ non-empty open $\Omega \subset B_r$,

$$\lambda_{\min}(\Omega) := \inf_{f \in \mathcal{F}(\Omega)\setminus\{0\}} \frac{\mathcal{E}(f,f)}{\|f\|_2^2} \geq \frac{c}{r^\beta}(\frac{\mu(B_r)}{\mu(\Omega)})^\nu,$$

where $\mathcal{F}(\Omega) := \{f \in \mathcal{F} : f = 0 \text{ in } X \setminus \Omega\}$.

Theorem 8.2 ([36], Theorem 12.1) *Assume* (VD). *Then.*

$$(UHK(\beta)) \Leftrightarrow (DUHK(\beta)) + (E(\beta)) \Leftrightarrow (FK(\beta)) + (E(\beta)).$$

Cf. $\beta = 2$ case for Riemannian manifolds ([38] Proposition 5.2)

$$(UHK(2) \Leftrightarrow (DUHK(2)) \Leftrightarrow (FK(2)).$$

9.4 Time derivative

First, we show the following well-known fact in the semigroup theory.

Lemma 9.7 *For any* $f \in L^2$*, let* $u_t := P_t f$*. Then,*

$$\|\partial_t u_t\|_2 \le \frac{1}{t-s}\|u_s\|_2, \qquad 0 < \forall s < t.$$

PROOF. Let $\{E_\lambda\}_{\lambda \ge 0}$ be spectral resolution of the operator $-\Delta$. Then

$$u_t = e^{t\Delta} f = \int_0^\infty e^{-t\lambda} dE_\lambda f, \quad \|u_t\|_2^2 = \int_0^\infty e^{-2t\lambda} d\|E_\lambda f\|^2.$$

Thus,

$$\partial_t u_t = \int_0^\infty (-\lambda) e^{-t\lambda} dE_\lambda f, \quad \|\partial_t u_t\|_2^2 = \int_0^\infty \lambda^2 e^{-2(t-s)\lambda} e^{-2s\lambda} d\|E_\lambda f\|^2.$$

Since $\lambda e^{-(t-s)\lambda} \le (t-s)^{-1}$, we obtain

$$\|\partial_t u_t\|_2^2 \le \frac{1}{(t-s)^2} \int_0^\infty e^{-2s\lambda} d\|E_\lambda f\|^2 = \frac{1}{(t-s)^2}\|u_s\|_2^2. \quad \square$$

Corollary 9.8 *For $t > 0$ and $z \in X$, $t \mapsto p_t(\cdot, z)$ is Frechet differentiable in L^2 and*

$$\|\partial_t p_t(\cdot, z)\|_2 \le \frac{1}{t-s}\sqrt{p_{2s}(z,z)}, \qquad 0 < \forall s < t.$$

PROOF. Let $f = p_\varepsilon(\cdot, z)$ for $\exists \varepsilon > 0$. Then, $u_t = P_t f = p_{t+\varepsilon}(\cdot, z)$. By Lemma 9.7,

$$\|\partial_t p_{t+\varepsilon}(\cdot, z)\|_2 \le \frac{1}{t-s}\|p_{s+\varepsilon}(\cdot, z)\|_2 = \frac{1}{t-s}\sqrt{p_{2(s+\varepsilon)}(z,z)}.$$

Replacing $t+\varepsilon, s+\varepsilon$ by t, s respectively, we obtain the result. □

Proposition 9.9 *For any $x, y \in X$, $t \mapsto p_t(x, y)$ is differentiable in $t > 0$ and*

$$|\frac{\partial_t}{\partial t} p_t(x,y)| \le \frac{2}{t}\sqrt{p_{t/2}(x,x)p_{t/2}(y,y)}.$$

PROOF. By the Chapman-Kolmogorov eq., $p_t(x,y) = (p_{t-s}(\cdot, x), p_s(\cdot, y))$, $\forall s \in (0,t)$, so $\partial_t p_t(x,y) = (\partial_t p_{t-s}(\cdot, x), p_s(\cdot, y))$. Applying Corollary 9.8,

$$|\frac{\partial_t}{\partial t} p_t(x,y)| \le \|\partial_t p_{t-s}(\cdot, x)\|_2 \|p_s(\cdot, y)\|_2 \le \frac{1}{t-s-r}\sqrt{p_{2r}(x,x)p_{2s}(y,y)},$$

$0 < \forall r < t-s$. Taking $s = r = t/4$, we obtain the result. □

7 Some open problems

- Simpler stable equivalence conditions for (PHI(Ψ)): It is not easy to check (CS(Ψ)) in examples. Quite recently, Barlow-Bass proved (PHI(β)) $\Leftrightarrow$ (VD) + (PI(β)) + (E(β)) for weighted graphs. Conjecture: (PHI(β)) $\Leftrightarrow$ (VD) + (PI(β)) + (RES(β)).

- Stability of (EHI): Is (EHI) stable under rough isometries?

- Stability of ($UHK(\Psi)$): Is ($UHK(\Psi)$) stable under rough isometries?

  Related conjecture by Grigor'yan: ($UHK(\beta)$) $\Leftrightarrow$ ($FK(\beta)$) + (Anti FK(β)), which guarantees the optimality of ($FK(\beta)$) for balls.

- RW on IIC on $\mathbb{Z}^d$: HK estimates for RW on infinite incipient clusters on $\mathbb{Z}^d$?

  $d = 2$ and d large enough, RW on such IIC is in the framework of resistance forms.

  So we have reasonable analytic estimates. Probabilistic estimates??