6 Application: RW on critical branching processes

RW on the percolation cluster on $\mathbb{Z}^d$ $(d \geq 2)$

Supercritical

De Masi, Ferrari, Goldstein and Wick (1989 [33]): Inv. principle for the annealed case

Sidoravicius and A.-S. Sznitman (2004 [74]): Inv. principle for the quenched case

Mathieu and Remy (2004): Isoperimetric ineq. and heat kernel decay

Barlow (2004 [5]): Detailed Gaussian heat kernel estimates

Critical Unknown!!

Kesten (1986 [55]): $d = 2$ 'subdiffusive behaviour'

cf. $d = 2$: Smirnov, Lawler, Schramn and Werner

$\Rightarrow$ Shape of the cont. limit etc. (Very Active)

6.2 The model and main results

$\mathcal{G}$: random tree. We could regard this in two ways.

- Critical percolation on the n_0-ary tree $\mathbb{B}$, condi. the cluster containing 0 being infinite
- Critical branching process with $Bin(n_0, 1/n_0)$ offspring distrib., condi. on non-extinction.

$\mathbb{B}$: n_0-ary tree, 0: the root, $E(\mathbb{B})$: edge set.

$\mathbb{B}_n$: the set of n_0^n points in the nth generation, $\mathbb{B}_{\le n} = \cup_{i=0}^n \mathbb{B}_i$.

η_e, $e \in E(\mathbb{B})$, be i.i.d. Bernoulli $1/n_0$ r.v. ($\eta_e = 1 \Leftrightarrow e$ is *open*.)

$$\mathcal{C}(0) := \{x \in \mathbb{B} : \text{ there exists an } \eta\text{–open path from 0 to } x\}$$

Clearly, $Z_n = |\mathcal{C}(0) \cap \mathbb{B}_n|$ is a critical GW process with $Bin(n_0, 1/n_0)$ offspring distri.

As Z has extinction probability 1, the cluster $\mathcal{C}(0)$ is P–a.s. finite.

Incipient infinite cluster (IIC) on $\mathbb{B}$. Two constructions.

Lemma 6.1 ([55], Lemma 1.14) *Let* $A \subset \mathbb{B}_{\leq k}$. *Then*

$$\lim_{n\to\infty} P(\mathcal{C}(0) \cap \mathbb{B}_{\leq k} = A | Z_n \neq 0) = |A \cap \mathbb{B}_k| P(\mathcal{C}(0) \cap \mathbb{B}_{\leq k} = A) =: \mathbb{P}_0(A).$$

$\exists 1 \mathbb{P}$*: extension of* $\mathbb{P}_0$ *to a prob. on the set of* ∞*-con. subsets of* $\mathbb{B}$ *containing* 0.

$\mathcal{G}'$: rooted labeled tree with the distri. $\mathbb{P} \Rightarrow$ IIC on $\mathbb{B}$. $\exists 1 H$ backbone of $\mathcal{G}'$.

(Another construction) $\{\xi_i\}_{i\geq 1}$: i.i.d., unif. distri. on $\{1, 2, \cdots, n_0\}$, indep. of (η_e).

For $n \geq 0$ let $\Xi_n = (0, \xi_1, \ldots, \xi_n)$, and let

$$\widetilde{\eta}_e := \begin{cases} 1 & \text{if } e = \{\Xi_n, \Xi_{n+1}\} \text{ for some } n \geq 0, \\ \eta_e & \text{otherwise,} \end{cases}$$

$$\mathcal{G} := \{x \in \mathbb{B} : \text{ there exists a } \widetilde{\eta}\text{–open path from } 0 \text{ to } x\},$$

$\mathcal{G}$ has law $\mathbb{P}$. $H = \{\Xi_n, n \geq 0\}$: backbone of $\mathcal{G}$

$$\text{Let} \qquad \mathbb{P}_x(\cdot) = \mathbb{P}(\cdot | x \in \mathcal{G}), \quad \mathbb{P}_{xy}(\cdot) = \mathbb{P}(\cdot | x, y \in \mathcal{G}),$$

$$\mathbb{P}_{x,y,b}(\cdot) = \mathbb{P}(\cdot | x, y \in \mathcal{G}, H = b).$$

For each fixed $\mathcal{G} = \mathcal{G}(\omega)$, $\{Y_t\}$: cont. time S.R.W. on $\mathcal{G}$,

P^x_ω: law of $\{Y_t\}$ starting at $x \in \mathcal{G}(\omega)$

E^x_ω: its average

$q^\omega_t(x, y) := \mathbb{P}^x(Y_t = y)/\mu_y$.

Kesten (1986 [55]): $\mathbb{P}$-distri. of $n^{-1/3} d(0, Y_n)$ converges.

Theorem 6.2 *(a)* $\exists c_0, c_1, c_2, S(x)$ *s.t.* $\mathbb{P}_x(S(x) \geq m) \leq c_0(\log m)^{-1}, \quad \forall x$ *and*

$$c_1 t^{-2/3}(\log\log t)^{-17} \leq q_t^\omega(x,x) \leq c_2 t^{-2/3}(\log\log t)^3 \qquad \forall t \geq S(x), x \in \mathcal{G}(\omega).$$

(b) $d_s(\mathcal{G}) := -2\lim_{t\to\infty} \frac{\log q_t^\omega(x,x)}{\log t} = 4/3\ \mathbb{P}-a.s.$

(c) $c_1 t^{-2/3} \leq \mathbb{E}_x[q_t^\cdot(x,x)] \leq c_2 t^{-2/3}$.

$q_t(x,x)$ does have oscillations of order $(\log\log t)^a$ as $t \to \infty$.

Proposition 6.3 $\liminf_{t\to\infty}(\log\log t)^{1/6} t^{2/3} q_{2t}^\omega(0,0) \leq 2, \qquad P_\omega^0 - a.s.$

Theorem 6.4 *(a)* $c_1 t^{1/3} \leq \mathbb{E}_x E_\omega^x d(x, Y_t) \leq \mathbb{E}_x E_\omega^x \sup_{0\leq s\leq t} d(x, Y_s) \leq c_2 t^{1/3}$.

(b) $\exists T(x)$ *with* $\mathbb{P}_x(T(x) < \infty) = 1$ *s.t.*

$$c_3 t^{1/3}(\log\log t)^{-12} \leq E_\omega^x[d(x, Y_t)] \leq c_4 t^{1/3}\log t \quad \forall t \geq T(x).$$

Quenched off-diagonal bounds for $q_t^\omega(x,y)$.

Theorem 6.5 *(1) Let $x, y \in \mathcal{G}$, $t > 0$ be s.t. $N := [\sqrt{d(x,y)^3/t}] \geq 8$.*

Then, $\exists F_* = F_*(x,y,t)$ with $\mathbb{P}_{x_0,y_0,b}(F_*(x,y,t)) \geq 1 - c_1 \exp(-c_2 N)$, s.t.

$$q_t^\omega(x,y) \leq c_3 t^{-2/3} \exp(-c_4 N), \qquad \forall \omega \in F_*.$$

(2) Let $x, y \in \mathcal{G}$, $m \geq 1$, $\kappa \geq 1$ and let $T = d(x,y)^3 \kappa / m^2$.

Then, $\exists G_* = G_*(x,y,m,\kappa)$ with $\mathbb{P}_{x,y,b}($ $G_*(x,y,m,\kappa)$ holds $) \geq 1 - c_1 \kappa^{-1}$, s.t.

$$q_{2T}(x,y) \geq c_2 T^{-2/3} e^{-c_3(\kappa + c_4)m}, \qquad \forall \omega \in G_*.$$

Annealed off-diagonal bounds for $q_t^\omega(x,y)$.

Theorem 6.6 *Let $x, y \in \mathbb{B}$. Then*

$$c_4 t^{-2/3} \exp(-c_5 (\frac{d(x,y)^3}{t})^{1/2}) \leq \mathbb{E}_{x,y} q_t^\omega(x,y) \leq c_1 t^{-2/3} \exp\big(- c_2 (\frac{d(x,y)^3}{t})^{1/2}\big),$$

where the lower bound is for $c_3 d(x,y) \leq t$.

Rescaled height process: $\widetilde{Z}^{(n)}_t = n^{-1/3} d(0, Y_{nt}), \quad t \geq 0.$

$\{Z^{(n)}\}$ are tight w.r.t. the annealed law $\mathbb{P}^* = \mathbb{P} \times P^0_\omega$. (Theorem 6.4 (a) or Kesten [55])

However, the large scale fluctuations in $\mathcal{G}$ mean that we do not have quenched tightness.

Theorem 6.7 *$\mathbb{P}$-a.s., the processes $(\widetilde{Z}^{(n)}, n \geq 1)$ are not tight with respect to P^0_ω.*

6.3 Ideas of the proof

Proof: analytic and probabilistic parts. Note: We cannot expect (VD)!!

Definition 6.8 *Let $x \in \mathcal{G}$, $r \geq 1$. Let $M(x, r)$ be the smallest number m s.t. $\exists A = \{z_1, \dots, z_m\}$ with $d(x, z_i) \in [r/4, 3r/4]$, for each i, so that any path γ from x to $B(x, r)^c$ must pass through the set A.*

Analytic estimates $B := B(x_0, r)$, $M := M(x_0, r)$, $V := V(x_0, r)$.

Proposition 6.9 *(a) (G, μ): weighted graph. Suppose that $\mu_{xy} \geq 1$ $\forall x \sim y$. Then*

$$q_{2rV(x,r)}(x, x) \leq \frac{2}{V(x, r)}, \quad x \in G,\ r > 0.$$

(b) G: tree. Let $V_1 = V_1(x_0, r) = V(x_0, r/(32M(x_0, r)))$. Then if $x \in B(x_0, r/(32M))$,

$$P^x(\tau_B \leq t) \leq (1 - \frac{V_1}{64MV}) + \frac{t}{2rV},$$

and

$$q_{2t}(x, x) \geq \frac{c_1 V_1(x_0, r)^2}{V(x_0, r)^3 M(x_0, r)^2} \quad \textit{for } t \leq \frac{rV_1(x_0, r)}{64M(x_0, r)}.$$

Proof. (a): similarly to Step A in subsection 5.2.

(b): similar argument as in Step B in subsection 5.2 (using the tree property and $M(x, r)$ instead of (VD)) gives the estimate of $E_\omega^x[\tau_{B(x,r)}]$. Then the argument in Step 3 in the proof of Proposition 4.1 gives the desired result.

Probabilistic estimates   On-diagonal estimates: Need information of $V(x,r)$ and $M(x,r)$!

The probability that $V(x,r)$ and $M(x,r)$ behave badly is ‘small’.

Proposition 6.10 *(a) Let $\lambda > 0$, $r \geq 1$, $x, y \in \mathbb{B}$, and b be a backbone. Then*

$$\mathbb{P}_{x,y,b}(V(x,r) > \lambda r^2) \leq c_0 \exp(-c_1 \lambda),$$

$$\mathbb{P}_{x,y,b}(V(x,r) < \lambda r^2) \leq c_2 \exp(-c_3/\sqrt{\lambda}).$$

(b) For any $\varepsilon > 0$

$$\limsup_{n\to\infty} \frac{V(0,n)}{n^2(\log\log n)^{1-\varepsilon}} = \infty, \quad \mathbb{P}-a.s.$$

(c) Let $r \geq 1$, $x, y \in \mathbb{B}$, and b be a backbone. Then

$$\mathbb{P}_{x,y,b}(M(x,r) \geq m) \leq c_4 e^{-c_5 m}.$$

These can be obtained, basically through large deviation estimates of the total population size of the critical branching process.

Idea of the proof of Proposition 6.10:

For simplicity, let $x \in H$, $d(0,x) > r$.

$|B(x,r)| \le V(x,r) \le 2|B(x,r)|$, so consider $|B(x,r)|$.

$\{\tilde{X}_n\}$: $\tilde{X}_0 = 1$, $\tilde{X}_1 \stackrel{(d)}{=} Bin(n_0 - 1, 1/n_0)$, from the 2nd generation, $Bin(n_0, 1/n_0)$.

$\tilde{Y}_n := \sum_{k=0}^{n} \tilde{X}_k$. Then,

$$\tilde{Y}_{r/2}[r/2] \stackrel{(d)}{\le} |B(x,r)| \stackrel{(d)}{\le} \tilde{Y}_r[r] + \tilde{Y}'_r[r].$$

(Here, for r.v. ξ, $\xi[n] :\stackrel{(d)}{=} \sum_{i=1}^{n} \xi_i$, where $\{\xi_i\}$ i.i.d. with $\xi_i \stackrel{(d)}{=} \xi$.)

Now let $Y_n := \sum_{k=0}^{n} X_k$: total population size up to generation n. Then,

$$P(Y_n[n] \ge \lambda n^2) \le c \exp(-c'\lambda), \quad P(Y_n[n] \le \lambda n^2) \le c \exp(-c'/\sqrt{\lambda}).$$

Similar estimates hold for $\tilde{Y}_n[n]$. $\quad \Rightarrow$ (a) holds.

We now define a 'good' random set.

Definition 6.11 *Let $x \in \mathbb{B}$, $r \geq 1$, $\lambda \geq 64$. $B(x,r)$ is λ–good if*

$$x \in \mathcal{G},\ \ r^2\lambda^{-2} \leq V(x,r) \leq r^2\lambda,\ \ M(x,r) \leq \frac{1}{64}\lambda,$$

$$V(x, r/\lambda) \geq r^2\lambda^{-4}, \quad \textit{and} \ \ V(x, r/\lambda^2) \geq r^2\lambda^{-6}.$$

By Proposition 6.10, we have the following.

Corollary 6.12 *For $x \in \mathbb{B}$ and any possible backbone b*

$$\mathbb{P}_{x,b}(B(x,r) \ \textit{is not}\ \lambda\text{–}\textit{good}) \leq c_1 e^{-c_2\lambda}.$$

By Prop 6.9, if $B(x,r)$ is λ-good, then

$$c_1' t^{-2/3}\lambda^{-17} \leq q_{2t}(x,x) \leq c_2' t^{-2/3}\lambda^3, \quad \frac{r^3}{\lambda^6} \leq \forall t \leq \frac{r^3}{\lambda^5}. \qquad (++)$$

Idea of the proof of Theorem 6.2. (a) Take $\lambda_n = e + (2/c_2)\log n$, $\quad r_n : r_n^3/\lambda_n^6 = e^n$, and let $F_n := \{B(x, r_n) \text{ is } \lambda_n\text{-good}\}$. By Cor 6.12, $\mathbb{P}(F_n^c) \le c/n^2$.

$N := \min\{m : F_n^c \text{ occurs } \exists n \ge m\}$. Then $\mathbb{P}(N \ge m) \le \sum_{n=m}^{\infty} \mathbb{P}(F_n^c) \le c/m$.

Let $S(x) := e^N$. By (++),

$$c_1' t^{-2/3}\lambda_n^{-17} \le q_{2t}(x,x) \le c_2' t^{-2/3}\lambda_n^3, \qquad \forall n \ge \log S(x) + 1,\ e^n \le t \le \lambda_n e^n. \qquad (*)$$

Take $n = n(t)$ s.t. $\log t \in [n(t) - 1, n(t)]$.

Then $(*)$ holds for $t \ge S(x)$ with $\lambda_{n(t)} \sim (2/c_2)\log\log t$. $\Rightarrow$ Thm 6.2 (a).

(b) $\lambda_n = n$, $r_n : r_n^3/\lambda_n^6 = t$. F_n as above. $N(\omega) := \min\{n : \omega \in F_n\}$.

By Cor 6.12, $\mathbb{P}(N > n) = \mathbb{P}(F_n^c) \le e^{-cn}$. Thus, using (++),

$$\mathbb{E}_x[q_t^{\cdot}(x,x)] \le c t^{-2/3}\mathbb{E}_x N^3 \le c' t^{-2/3}.$$

Lower bound is easy by (++) and Cor 6.12. $\square$

To get off-diagonal estimates, we need to take more refined 'good' random sets.

7 Some open problems

- Simpler stable equivalence conditions for (PHI(Ψ)): It is not easy to check (CS(Ψ)) in examples. Quite recently, Barlow-Bass proved (PHI(β)) $\Leftrightarrow$ (VD) + (PI(β)) + (E(β)) for weighted graphs. Conjecture: (PHI(β)) $\Leftrightarrow$ (VD) + (PI(β)) + (RES(β)).

- Stability of (EHI): Is (EHI) stable under rough isometries?

- Stability of ($UHK(\Psi)$): Is ($UHK(\Psi)$) stable under rough isometries?

  Related conjecture by Grigor'yan: ($UHK(\beta)$) $\Leftrightarrow$ ($FK(\beta)$) + (Anti FK(β)), which guarantees the optimality of ($FK(\beta)$) for balls.

- RW on IIC on $\mathbb{Z}^d$: HK estimates for RW on infinite incipient clusters on $\mathbb{Z}^d$?

  $d = 2$ and d large enough, RW on such IIC is in the framework of resistance forms.

  So we have reasonable analytic estimates. Probabilistic estimates??