対称拡散過程の熱核評価、ハルナック不等式の安定性とその応用

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X: locally compact separable metric space (diam $X = \infty$)

$(\mathcal{E}, \mathcal{F})$: reg. local Dirichlet form on $L^2(X, \mu)$

$-\Delta$, $\{X_t\}_t$: the corresponding non-neg. S.A. operator and the diffusion.

- Elliptic Harnack inequality (EHI): $\exists c_3 > 0$ s.t. $\forall B(x, R)$,

$\forall u$: non-negative harmonic fu. on $B(x, R)$ (i.e. $\Delta u(x) = 0$ for $x \in B(x, R)$), then

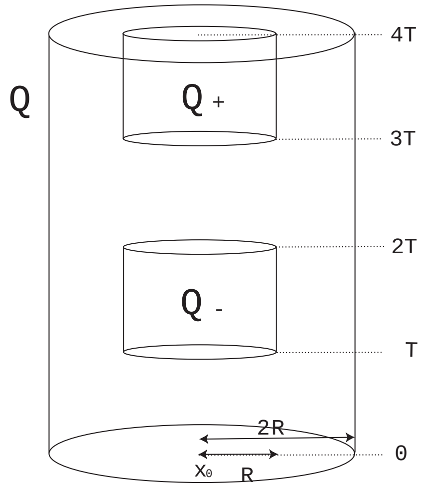
$$\sup_{B(x,R/2)} u \le c_3 \inf_{B(x,R/2)} u. \tag{EHI}$$

Let $\beta \ge 2$ and denote $V(x, R) := \mu(B(x, r))$.

- (Sub-)Gaussian heat kernel estimates:

$$\frac{c_4}{\mu(B(x, t^{1/2}))} \exp(-\frac{d(x, y)^2}{c_4 t}) \le p_t(x, y) \le \frac{c_5}{\mu(B(x, t^{1/2}))} \exp(-\frac{d(x, y)^2}{c_5 t}). \tag{HK(2)}$$

$$\frac{c_4}{\mu(B(x, t^{1/\beta}))} \exp(-(\frac{d(x, y)^\beta}{c_4 t})^{\frac{1}{\beta-1}}) \le p_t(x, y) \le \frac{c_5}{\mu(B(x, t^{1/\beta}))} \exp(-(\frac{d(x, y)^\beta}{c_5 t})^{\frac{1}{\beta-1}}). \tag{HK(β)}$$



- Let $Q = Q(x_0, T, R) = (0, 4T) \times B(x_0, 2R)$,

$$Q_-(T, 2T) \times B(x_0, R) \ \text{ and } \ Q_+ = (3T, 4T) \times B(x_0, R).$$

Parabolic Harnack inequality (PHI(β)): $\exists c_6 > 0$ s.t. the following holds.

Let $x_0 \in X$, $R > 0$, $T = R^\beta$, and $u = u(t,x) : Q \to \mathbb{R}_+$ satisfies $\frac{\partial u}{\partial t} = \Delta u$ in Q. Then,

$$\sup_{Q_-} u \leq c_6 \inf_{Q_+} u. \tag{PHI(β)}$$

$(HK(\beta)) \Leftrightarrow (\mathrm{PHI}(\beta))$ から拡散過程の様々な性質が導き出せる

- $c_1 t^{1/\beta} \le E^x[d(x, X_t)] \le c_2 t^{1/\beta}$ ($\beta > 2$: 劣拡散的)
- 重複対数の定理　(i.e. $\limsup_{t\to\infty} \frac{d(X_t, X_0)}{t^{1/\beta}(\log\log t)^{1-1/\beta}} = C$, P^x-a.s.)
- 熱方程式の解のHölder連続性
- 楕円型ハルナック不等式 (EHI)
- Liouville property (i.e. positive harm. fu. on X is const.)

  Indeed, if $m_u := \inf_X u$, then by (EHI), $\sup_B(u - m_u) \le c \inf_B(u - m_u) \to 0$ as $B \to \infty$. So $u \equiv m_u$, μ-a.e.
- グリーン核の評価

<u>歴史</u>

Divergence form $\mathcal{L} = \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}(a_{ij}(x)\frac{\partial}{\partial x_j})$ on $\mathbb{R}^n$ satisfying the unif. ellip. cond. (i.e. $\sigma^{-1}I \le a(\cdot) \le \sigma I$ for $\exists \sigma \ge 1$).

- De Giorgi ('57), Nash ('58) [70]: Hölder cont. for elliptic/parabolic functions
- Moser ('61,'64,'71) [69,68,67]: Harnack ineq.
- Aronson ('67) [2]: (HK(2))
- Krylov-Safanov ('80): Prob. proof for Harnack
- Davies ('81, '87, '89) [32,31]: Off-diagonal upper estimates
- Fabes-Stroock ('86) [34]: A new proof of Moser's PHI using the old idea of Nash.
- Carlen-Kusuoka-Stroock ('87) [25]: equiv. of the Nash inequalities
- Li-Yau ('86) [65]: smooth non-cpt compl. R-mfd, non-neg. Ricci, $\Delta \Rightarrow$ (HK(2))

- Grigor'yan ('92) [39], Saloff-Coste ('92) [73]: (HK(2)) $\Leftrightarrow$ (VD) + (PI(2))

- Biroli-Mosco ('95) [20], Sturm ('95,'96) [75,76], Delmotte ('99) [32]: extension to Dirichlet forms on meas. met. spaces and graphs

(A) Volume doubling (VD): $\quad V(x, 2R)) \le c_1 V(x, R), \quad \forall x \in X,\ R \ge 0.$

(B) Poincaré inequality (PI(β)): $\exists c_2$ s.t. $\forall B = B(x, R) \subset X$ and $\forall f \in \mathcal{F}$,

$$\int_B (f(x) - \overline{f}_B)^2 d\mu(x) \le c_2 R^\beta \mathcal{E}_B(f, f), \quad \text{where } \overline{f}_B = \frac{1}{\mu(B)} \int_B f(x) d\mu(x). \qquad \text{(PI}(\beta))$$

Sub-Gaussian case

- Grigor'yan-Telcs ('01,'02) [42,41], Barlow-Bass ('03) [9]

- Kigami ('04) [57], Grigor'yan ('05) [36]

- Barlow-Coulhon-K ('05) [15], Barlow-Bass-K ('05) [14]

講演プラン

1. ガウス型の場合の古典的手法
2. 測度付き距離空間、グラフ上のディリクレ形式：ハルナック不等式と熱核評価
3. 強再帰的な場合
4. 臨界点における確率モデルの熱核評価

2.1 The Nash inequality

X: locally compact separable metric space

$(\mathcal{E}, \mathcal{F})$: Dirichlet form on $L^2(X, \mu)$

$-\Delta$, $\{P_t\}$: the corresponding non-negative self-adjoint operator and the semigroup

Theorem 2.1 (The Nash inequality, [25])

The following are equivalent for any $\delta > 0$.

1) There exist $c_1, \theta > 0$ *such that for all* $f \in \mathcal{F} \cap L^1$,

$$\|f\|_2^{2+4/\theta} \le c_1(\mathcal{E}(f,f) + \delta\|f\|_2^2)\|f\|_1^{4/\theta}, \qquad (Nash)$$

where $\|f\|_p := (\int_X |f|^p d\mu)^{1/p}$.

2) $\forall t > 0$, $P_t(L^1) \subset L^\infty$ *and it is a bounded operator. Moreover,* $\exists c_2, \theta > 0$ *s.t.*

$$\|P_t\|_{1\to\infty} \le c_2 e^{\delta t} t^{-\theta/2}, \qquad \forall t > 0.$$

Proof of Theorem 2.1:

1) $\Rightarrow$ 2) : Let $f \in L^2 \cap L^1$ with $\|f\|_1 = 1$ and $u(t) := (P_t f, P_t f)_2$. Then,

$$\begin{aligned}\frac{u(t+h)-u(t)}{h} &= \frac{1}{h}(P_{t+h}f + P_t f, P_{t+h}f - P_t f)_2 = (P_{t+h}f + P_t f, \frac{(P_h - I)P_t f}{h})_2 \\ &\xrightarrow{h\downarrow 0} 2(P_t f, \Delta P_t f)_2 = -2\mathcal{E}(P_t f, P_t f).\end{aligned}$$

Hence $u'(t) = -2\mathcal{E}(P_t f.P_t f)$. Now by 1),

$$2u(t)^{1+2/\theta} \leq c_1(-u'(t) + 2\delta u(t))\|P_t f\|_1^{4/\theta} \leq c_1(-u'(t) + 2\delta u(t)),$$

because $\|P_t f\|_1 \leq \|f\|_1 = 1$. Thus,

$$2(e^{-2\delta t}u(t))^{1+2/\theta} \leq 2e^{-2\delta t}u(t)^{1+2/\theta} \leq -c_1(e^{-2\delta t}u(t))'.$$

Set $v(t) = (e^{-2\delta t}u(t))^{-2/\theta}$, then $v'(t) \geq 4/(c_1\theta)$. Since $\lim_{t\downarrow 0} v(t) = u(0)^{-2/\theta} > 0$, it follows that $v(t) \geq 4t/(c_1\theta)$. This means $u(t) \leq c_2 e^{2\delta t} t^{-\theta/2}$ where $c_2 = (c_1\theta/4)^{\theta/2}$.

Hence

$$\|P_t f\|_2 \le c_3 e^{\delta t} t^{-\theta/4} \|f\|_1, \qquad \forall f \in L^2 \cap L^1,$$

which implies $\|P_t\|_{1\to 2} \le c_3 e^{\delta t} t^{-\theta/4}$. Since $P_t = P_{t/2} \circ P_{t/2}$ and $\|P_{t/2}\|_{1\to 2} = \|P_{t/2}\|_{2\to\infty}$, we obtain 2). $\square$

Remark. Generalization of Theorem 2.1: by Coulhon [28], Tomisaki [77] etc. See subsection 8.1.

$$\hat{\mathcal{F}} := \{h + c : h \in \mathcal{F}_b, c \in \mathbb{R}\}$$

$$\hat{\mathcal{F}}_\infty := \{\psi \in \hat{\mathcal{F}} : e^{-2\psi}\Gamma(e^{\psi},\, e^{\psi}) \ll \mu, e^{2\psi}\Gamma(e^{-\psi},\, e^{-\psi}) \ll \mu\}.$$

Theorem 2.4 (Carlen-Kusuoka-Stroock [25], Theorem 3.25)

Assume (Nash). *Then,* $\exists c > 0$ *s.t.* $\forall \rho \in (0, 1]$,

$$p_t(x, y) \le c\,(\rho t)^{-\theta/2} e^{-E((1+\rho)t,x,y)+\delta\rho t} \qquad \textit{for } t > 0 \textit{ and } x, y \in X, \tag{2.4}$$

where

$$E(t, x, y) := \sup\{|\psi(x) - \psi(y)| - t\Lambda(\psi)^2 :\ \Lambda(\psi) < \infty\}$$

with

$$\Lambda(\psi)^2 := \max\left\{\|\frac{d\, e^{-2\psi}\Gamma(e^{\psi},\, e^{\psi})}{d\mu}\|_\infty,\ \|\frac{d\, e^{2\psi}\Gamma(e^{-\psi},\, e^{-\psi})}{d\mu}\|_\infty.\right\}.$$

証明の方針： Step I　以下の式を$\forall f \in \hat{\mathcal{F}}, \forall p \in [1,\infty)$で示す([25], Theorem 3.9)。

$$\mathcal{E}(e^{\psi} f^{2p-1}, e^{-\psi} f) \geq p^{-1}\mathcal{E}(f^p, f^p) - 9p\Lambda(\psi)^2 \|f\|_{2p}^{2p}.$$

Step II: $f_t(x) := e^{\psi(x)}[P_t(e^{-\psi} f)](x)$ とし、上の不等式と(Nash)を以下の式に用いる。

$$\frac{\partial}{\partial t}\|f_t\|_{2p}^{2p} = -2p\mathcal{E}(e^{\psi} f_t^{2p-1}, e^{-\psi} f_t).$$

Step III: 得られた微分不等式を評価する([25], Lemma 3.21)。 □

<u>Upper boundの出し方</u>　$\mathcal{L} = \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}(a_{ij}(x)\frac{\partial}{\partial x_j})$ on $\mathbb{R}^n$ satisfying $\sigma^{-1} I \leq a(\cdot) \leq \sigma I$ for $\exists \sigma \geq 1$. In this case, (Nash) holds with $\theta = n$, $\delta = 0$ and

$$\Lambda(\psi)^2 = \sup_x (\nabla\psi(x), a(x)\nabla\psi(x)).$$

Let $\rho = 1$. Taking $\psi(x) = \theta \cdot x$ for some $\theta \in \mathbb{R}^n$ in (2.4), we get

$$p_t(x,y) \leq c_1 t^{-d/2} \exp(\theta \cdot (x-y) + 2\|\theta\|^2 \sigma t).$$

Optimize: $\theta = (y-x)/(4\sigma t)$, we obtain

$$p_t(x,y) \le c_1 t^{-d/2} \exp(-\frac{|y-x|^2}{8\sigma t}),$$

and the Gaussian upper bound is obtained.

<u>実はもっと良い評価が出せる</u>

$$d_{\mathcal{E}}(x,y) := \sup\{\psi(x)-\psi(y) : \psi \in \hat{\mathcal{F}}_\infty \cap C(X), \Lambda(\psi) \le 1\}.$$

This is a metric and sometimes called an *intrinsic metric*. By a simple computation,

$$E((1+\rho)t, x, y) = \frac{d_{\mathcal{E}}(x,y)^2}{4(1+\rho)t}.$$

So, we conclude

$$p_t(x,y) \le c_1 (\rho t)^{-d/2} \exp(-\frac{d_{\mathcal{E}}(x,y)^2}{4(1+\rho)t}).$$

Remark. For $\beta > 2$, this method does not work! Indeed, for diffusions on 'typical' fractals, the energy meas. is singular to the Hdff. measure ([47,61]) so $d_{\mathcal{E}}(x,y) \equiv 0$.

2.3 Moser's arguments

X: Riemannian manifold

Δ: the Laplace-Beltrami operator satisfying (PI(β)).

μ the Riemannian measure satisfying $c_1 r^\alpha \le \mu(B(x,r)) \le c_2 r^\alpha$, $\forall x \in X$, $r \ge 1$.

$$\fint_B f = \mu(B)^{-1} \int_B f d\mu.$$

(PI(β)) $\Rightarrow$ (2.5): the Sobolev inequality

$$\Big(\fint_B |f|^{2\kappa}\Big)^{1/\kappa} \le c_1 R^\beta \fint_B |\nabla f|^2, \qquad f \in C_0^\infty(B). \tag{2.5}$$

Here $\kappa = \bar{\alpha}/(\bar{\alpha}-2)$, $\bar{\alpha} = 3 \vee \alpha$.

$d\Gamma(f,f) = |\nabla f|^2 d\mu$ for $f \in \mathcal{F}$.

Let $u > 0$ be harmonic on B, $v = u^p$ for $p > 0$, $1/2 < a_2 < a_1 < 1$, $B_i := B(x_0, a_i R)$.

$\varphi \in C_0^\infty(B_1)$: a cut-off function for $B_2 \subset B_1$.

By "converse to the Poincaré inequality" (see Lemma 4.6 below),

$$\int_{B_1} |\varphi \nabla v|^2 \le c_2 \|\nabla \varphi\|_\infty^2 \int_{B_1} v^2. \tag{2.6}$$

Using (2.5) with $f = v$ and (2.6),

$$(\fint_{B_2} u^{2\kappa p})^{1/\kappa} \le c_3 R^\beta \fint_{B_2} |\nabla v|^2 \le c_3 R^\beta \fint_{B_1} \varphi^2 |\nabla v|^2 \le c_4 R^\beta \|\nabla \varphi\|_\infty^2 \int_{B_1} v^2.$$

Take "classical" cut-off function $\varphi(x) = \frac{d(x,B^c)}{R(a_1-a_2)} \quad \Rightarrow \quad \|\nabla \varphi\|_\infty^2 \le \frac{c_5}{(a_1-a_2)^2 R^2}$. Thus

$$(\fint_{B_2} u^{2\kappa p})^{1/\kappa} \le c_6 R^{\beta-2} (a_1 - a_2)^{-2} \fint_{B_1} u^{2p}. \tag{2.7}$$

Let $a_k := (1+2^{-k})/2, p_k := p\kappa^k$ and $B_k := B(x_0, a_k R)$. (Then $a_k - a_{k+1} = 2^{-k-2}$.)

Set $I_k := (\fint_{B_{k+1}} u^{2p_k})^{1/(2p_k)}$. Then, by (2.7) we have

$$I_{k+1} \le (c_7 R^{\beta-2} 2^{2k})^{1/(2p_k)} I_k.$$

By iteration (this part is the first part of Moser's argument), we have

$$I_k \le \Pi_{l=0}^{k-1} (c_7 R^{\beta-2} 2^{2l})^{1/(2p_l)} I_0 \le c_8 R^{c'(\beta-2)} I_0.$$

The last inequality is due to $\sum_l \kappa^{-l} < \infty$ and $\sum_l l\kappa^{-l} < \infty$, because $\kappa > 1$.

Take $k \to \infty$. Since $p_k \to \infty$ and u is continuous, we have

$$\sup_{y\in B(x_0,R/2)} u(y) \le c_8 R^{c'(\beta-2)} \Big(\fint_B u^{2p}\Big)^{1/(2p)} =: c_8 R^{c'(\beta-2)} \Phi(2p, B).$$

Taking u^{-1} instead of u, we have

$$\inf_{y\in B(x_0,R/2)} u(y) \ge c_8' R^{-c'(\beta-2)} \Phi(-2p, B).$$

Now, let $\beta = 2$. (The second part of Moser's argument; comparison between $\Phi(2p, B)$ and $\Phi(-2p, B)$.) Let $w := \log u$.

A) $\int_Q |\nabla w|^2 \le c\mu(Q)/R^2$ (Prop 4.9 (a)).

B) (The John-Nirenberg ineq. (Exp. integrability of BMO functions).)

Q_0: a cube. If $f \in L^1(Q_0)$ satisfies $\fint_Q |f - f_Q| \le 1$, $\forall Q \subset Q_0$ (such functions are called BMO fu.), then $\exists c, c' > 0$ s.t. $\fint_{Q_0} \exp(cf) \le c'$.

Using Schwarz, (PI(2)) and (A),

$$\Big(\fint_Q |w - w_Q|\Big)^2 \le \fint_Q |w - w_Q|^2 \le c(R^2/\mu(Q)) \int_Q |\nabla w|^2 \le C.$$

So, applying (B), we obtain

$$\fint_B u^{q_0} = \fint_B \exp(q_0 w) \le c, \qquad \fint_B u^{-q_0} = \fint_B \exp(-q_0 w) \le c',$$

for some $q_0 > 0$. Taking $p = q_0/2$, we conclude

$$\sup_{B(x_0,R/2)} u \le c_1 \Phi(q_0, B) \le c_2 \Phi(-q_0, B) \le c_3 \inf_{B(x_0,R/2)} u \quad \Rightarrow \text{(EHI)}. \qquad \square$$

Remark. If $\beta > 2$, one still obtains an L^∞ bound on u in $B(x, R/2)$, but the constant now depends on R, so that the final constant in the (EHI) will also depend on R!

As we see, the problem arises in the first ('easy') part of Moser's argument. Instead of the linear cut-off functions, one needs cut-off functions such that the term $R^{\beta-2}$ in the right hand side of (2.7) disappears.

3 Framework and main theorem

3.1 Framework

Metric measure spaces (MM)

(X, d): connected loc. cpt compl. sep. metric space (d: geodesic)

μ: Borel measure on X s.t. $0 < \mu(B) < \infty, \forall B \neq \emptyset$

$B(x,r) = \{y : d(x,y) < r\}$, $V(x,r) = \mu(B(x,r))$.

For simplicity, assume diam $X = \infty$.

Metric measure Dirichlet spaces (MMD)

(X, d, μ): MM space, $(\mathcal{E}, \mathcal{F})$: regular, strong local Dirichlet form on $L^2(X, \mu)$

Δ: corresponding (non-positive) self-adjoint operator $(\mathcal{E}(h,g) = -\int \Delta h g \, d\mu)$

$\{P_t\}$: corresponding semigroup

Assume that $(\mathcal{E}, \mathcal{F})$ is conservative (i.e. $P_t 1 = 1, \ \forall t > 0$).

$\Gamma(f,g)$: signed measure

$\forall f \in \mathcal{F}_b$, $\exists 1 \Gamma(f,f)$: Borel measure (the *energy measure*) satisfying

$$\int_X g d\Gamma(f,f) = 2\mathcal{E}(f,fg) - \mathcal{E}(f^2,g), \qquad g \in \mathcal{F}_b.$$

(Rem: We take the quasi-continuous modification of $g \in \mathcal{F}_b$ without writing $\tilde{g}$.)

$$\Gamma(f,g) := \frac{1}{2}(\Gamma(f+g,f+g) - \Gamma(f,f) - \Gamma(g,g)), \qquad f,g \in \mathcal{F}.$$

Leibniz and chain rules: if $f_1,\ldots,f_m,g,\varphi(f_1,\ldots,f_m) \in \mathcal{F}_b$,

$$d\Gamma(fg,h) = f d\Gamma(g,h) + g d\Gamma(f,h),$$

$$d\Gamma(\varphi(f_1,\ldots,f_m),g) = \sum_{i=1}^{m} \frac{\partial \varphi}{\partial x_i}(f_1,\ldots,f_m) d\Gamma(f_i,g).$$

- $Y = (Y_t, t \geq 0, \mathbb{P}^x, x \in X\}$: diffusion process associated with $(\mathcal{E},\mathcal{F})$ on $L^2(X,\mu)$.

Examples. 1. X: Riemannian manifold, d: Riem. metric, μ: Riem. measure.

$\mathcal{C}$: C^∞ functions on X with compact support,

$$\mathcal{E}(f,f) = \int_X |\nabla f|^2 d\mu, \quad f \in \mathcal{C}.$$

$\mathcal{E}$: completion of $\mathcal{C}$ with respect to the norm $||f||_2 + \mathcal{E}(f,f)^{1/2}$, $d\Gamma(f,g) = \nabla f \cdot \nabla g \, d\mu$.

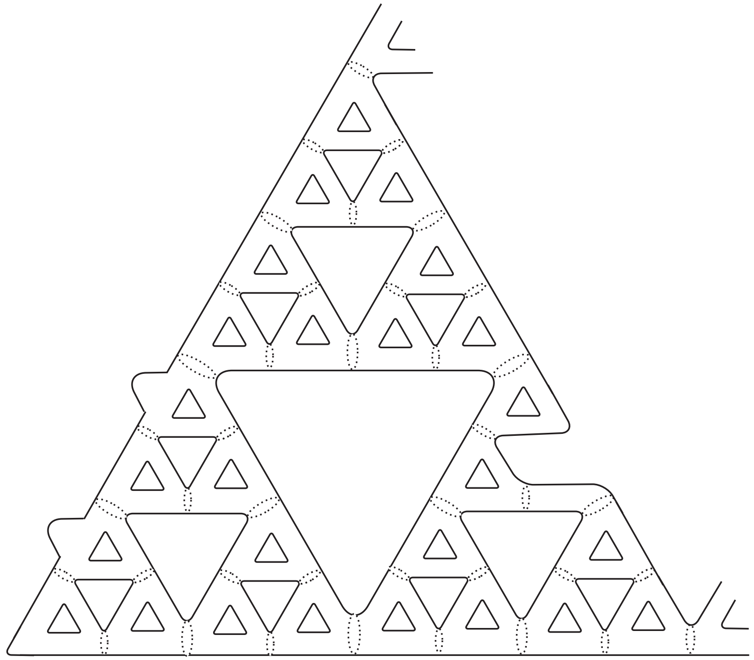
2. Cable system of a graph. (G, E, ν): a weighted graph

Define the *cable system* G_C by replacing each edge of G by a copy of $(0,1)$.

μ: measure on G_C given by $d\mu(t) = \nu_{xy} \, dt$

$\mathcal{C}$: the functions in $C(G_C)$ which have compact support and are C^1 on each cable

$$\mathcal{E}(f,f) = \int_{G_C} |f'(t)|^2 d\mu(t).$$



3. D: a domain in $\mathbb{R}^d$ with a smooth boundary

$\mathcal{C} := C_0^2(\overline{D})$, μ: Lebesgue measure, and

$$\mathcal{E}(f,f) = \frac{1}{2}\int_D |\nabla f|^2 d\mu.$$

The associated diffusion Y is Brownian motion on D with normal reflection on ∂D.

4. Diffusions on fractals. $F \subset \mathbb{R}^d$: connected set with diameter 1

Suppose $\exists d$ geodesic metric on F. μ: Hausdorff α-measure on F (with respect to d)

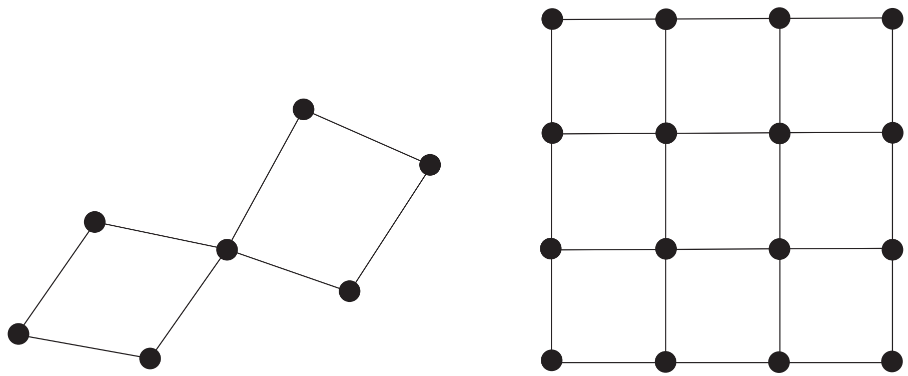
Suppose that $\mu(B(x,r)) \asymp r^\alpha$, $x \in F$, $r > 0$. Let

$$\begin{aligned} N_{\sigma,\infty}(f) &:= \sup_{0<r\le 1} r^{-\alpha-2\sigma} \int_F \int_F 1_{B(y,r)}(x)|f(x)-f(y)|^2 d\mu(x)d\mu(y), \\ \Lambda^\sigma_{2,\infty}(F) &:= \{u \in L^2(F,\mu) : N_{\sigma,\infty}(u) < \infty\}. \end{aligned}$$

There exist many fractals satisfying the above with a Dirichlet form $\mathcal{E}$ on $L^2(F,\mu)$ for which the domain $\mathcal{F}$ of $\mathcal{E}$ is given by $\Lambda^{\beta/2}_{2,\infty}(F)$, and $\mathcal{E}(f,f) \asymp N_{\sigma,\infty}(f)$.

$F = F_{SG}$: (compact) Sierpinski gasket, F_n: set of vertices of triangles of side 2^{-n}; $x \sim y \Leftrightarrow x$ and y are in some triangle of side 2^{-n}. Then, with $\beta = \log 5/\log 2$,

$$\mathcal{E}(f,f) = c \lim_{n\to\infty} (5/3)^n \sum_{x\sim y} (f(x)-f(y))^2, \qquad f \in \Lambda^{\beta/2}_{2,\infty}(F).$$



Weighted graphs (G,E): an infinite locally finite connected graph, $x \sim y \Leftrightarrow (x,y) \in E$.

$\{\mu_{xy}\}_{x,y\in G}$: edge weights (conductances) $\mu_{xy} = \mu_{yx} \geq 0$, $\mu_{xy} > 0 \Leftrightarrow x \sim y$.

μ: $\mu(A) := \sum_{x\in A} \mu_x$, where $\mu_x := \sum_y \mu_{xy}$, d: graph distance

(G,μ) has *controlled weights* (p_0-condition) if there exists $p_0 > 0$ such that

$$\frac{\mu_{xy}}{\mu_x} \geq p_0, \qquad \forall x \sim y \in G.$$

The Laplacian and the Dirichlet form are defined on (G, μ) by

$$\Delta f(x) = \frac{1}{\mu_x} \sum_y \mu_{xy}(f(y) - f(x)).$$

$$\mathcal{E}(f,g) = \tfrac{1}{2} \sum_x \sum_y (f(x) - f(y))(g(x) - g(y))\mu_{xy}, \quad f, g \in \mathcal{F} := L^2(G, \mu).$$

If $f \in \mathcal{F}$ we define the measure $\Gamma_G(f,f)$ on G by setting

$$\Gamma_G(f,f)(x) = \sum_{y \sim x} (f(x) - f(y))^2 \mu_{xy}.$$

• $Y = \{Y_t\}_{t \geq 0}$: continuous time RW on G associated with $\mathcal{E}$ and the measure μ.

Y is called the simple random walk on G if $\mu_{xy} \equiv 1$ for $x \sim y$.

Y waits at x for an exponential mean 1 random time and then moves to a neighbour y of x with probability proportional to μ_{xy}.

$q_t(\cdot,\cdot)$: the transition density (heat kernel density) of Y with respect to μ;

$$q_t(x,y) = \mathbb{P}^x(Y_t = y)/\mu_y. \tag{3.1}$$

3.2 Inequalities

$(X, d, \mu, \mathcal{E})$: MMD space

Let $\beta, \bar{\beta} \geq 2$ and

$$\Psi(s) = \Psi_{\bar{\beta},\beta}(s) = \begin{cases} s^{\bar{\beta}} & \text{if } s \leq 1 \\ s^{\beta} & \text{if } s > 1. \end{cases} \tag{3.1}$$

$\Psi(s)$ will give the space/time scaling on the space X.

(I) Volume doubling (VD):

$$V(x, 2R) \leq c_1 V(x, R), \qquad \forall x \in X,\ R \geq 0. \tag{VD}$$

(VD) implies that $\exists c_1, \alpha > 0$ s.t. if $x, y \in X$ and $0 < r < R$, then

$$\frac{V(x, R)}{V(y, r)} \leq c_1 \left(\frac{d(x, y) + R}{r}\right)^{\alpha}. \tag{9.1}$$

See subsection 9.1 for other consequences of (VD).

(II) Poincaré inequality (PI(Ψ)): $\exists c_2$ s.t. $\forall B = B(x,R) \subset X$ and $\forall f \in \mathcal{F}$,

$$\int_B (f(x) - \overline{f}_B)^2 d\mu(x) \le c_2 \Psi(R) \int_B d\Gamma(f,f), \tag{PI(Ψ)}$$

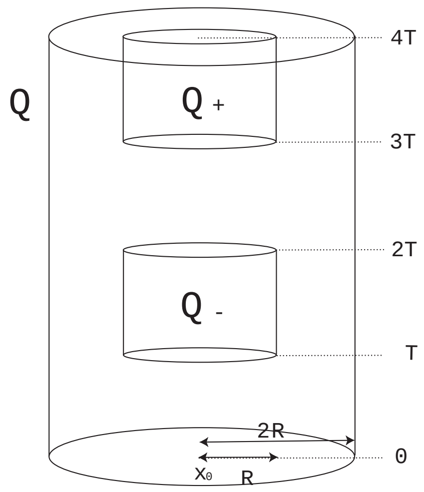
where $\overline{f}_B = \mu(B)^{-1} \int_B f(x) d\mu(x)$.

(III) u is *harmonic* on a domain D if $u \in \mathcal{F}_{loc}$ and $\mathcal{E}(u,g) = 0\ \forall g \in \mathcal{F}$ with support in D. ($u \in \mathcal{F}_{loc} \Leftrightarrow \forall$ G rel. compact open, $\exists w \in \mathcal{F}$ s.t. $u = w$ μ-a.e. on G.)

Elliptic Harnack inequality (EHI): $\exists c_3 > 0$ s.t. $\forall B(x,R)$, $\forall u$: non-negative harmonic function on $B(x,R)$, $\exists$ a quasi-continuous modification $\tilde{u}$ of u that satisfies

$$\sup_{B(x,R/2)} \tilde{u} \le c_3 \inf_{B(x,R/2)} \tilde{u}. \tag{EHI}$$

Remark. A standard argument (see subsec. 9.3), (EHI) implies $\tilde{u}$ is Hölder continuous.



(IV) Let $Q = Q(x_0, T, R) = (0, 4T) \times B(x_0, 2R)$,

$$Q_-(T, 2T) \times B(x_0, R) \;\text{ and }\; Q_+ = (3T, 4T) \times B(x_0, R).$$

Parabolic Harnack inequality (PHI(Ψ)): $\exists c_4 > 0$ s.t. the following holds.

Let $x_0 \in X$, $R > 0$, $T = \Psi(R)$, and $u = u(t, x) : Q \to \mathbb{R}_+$ satisfies $\frac{\partial u}{\partial t} = \Delta u$ in Q.

$\exists$ a quasi-continuous modification $\tilde{u}$ of u (for each t) that satisfies

$$\sup_{Q_-} \tilde{u} \le c_4 \inf_{Q_+} \tilde{u}. \tag{PHI(Ψ)}$$

(V) A, B: disjoint subsets of X. We define the effective resistance $R(A,B)$ by

$$R(A,B)^{-1} = \inf\{\int_X d\Gamma(f,f) : f = 0 \text{ on } A \text{ and } f = 1 \text{ on B}, \ f \in \mathcal{F}\}. \tag{3.4}$$

(RES(Ψ)): $\exists c_1, c_2 > 0$ s.t. $\forall x_0 \in X$, $\forall R \geq 0$,

$$c_1 \frac{\Psi(R)}{V(x_0,R)} \leq R(B(x_0,R), B(x_0,2R)^c) \leq c_2 \frac{\Psi(R)}{V(x_0,R)}. \tag{RES(Ψ)}$$

(VI) (CS(Ψ)): $\exists \theta \in (0,1]$, $\exists c_1, c_2 > 0$ s.t. the following holds.

$\forall x_0 \in X$, $\forall R > 0$, $\exists$ a cut-off function $\varphi (= \varphi_{x_0,R})$ with the properties:

(a) $\varphi(x) \geq 1$ for $x \in B(x_0, R/2)$. (b) $\varphi(x) = 0$ for $x \in B(x_0,R)^c$.

(c) $|\varphi(x) - \varphi(y)| \leq c_1 (d(x,y)/R)^\theta$, $\forall x, y \in X$.

(d) For any ball $B(x,s)$ with $0 < s \leq R$ and $f \in \mathcal{F}$,

$$\int_{B(x,s)} f^2 d\Gamma(\varphi,\varphi) \leq c_2 (s/R)^{2\theta} (\int_{B(x,2s)} d\Gamma(f,f) + \Psi(s)^{-1} \int_{B(x,2s)} f^2 d\mu). \tag{3.5}$$

(VII) For $(t,r)\in(0,\infty)\times[0,\infty)$, let

$$\Lambda_1=\{(t,r): t\le 1\vee r\},\quad \Lambda_2=\{(t,r): t\ge 1\vee r\},\quad g_\beta(r,t)=\exp\big(-(\frac{r^\beta}{t})^{1/(\beta-1)}\big).$$

(HK(Ψ)): the heat kernel $p_t(x,y)$, $x,y\in X$ and $t\in(0,\infty)$, exists and satisfies

$$\frac{c_1 g_{\bar\beta}(c_2 d(x,y),t)}{V(x,t^{1/\bar\beta})}\le p_t(x,y)\le\frac{c_3 g_{\bar\beta}(c_4 d(x,y),t)}{V(x,t^{1/\bar\beta})},\qquad \forall (t,d(x,y))\in\Lambda_1, \tag{3.6}$$

$$\frac{c_1 g_{\beta}(c_2 d(x,y),t)}{V(x,t^{1/\beta})}\le p_t(x,y)\le\frac{c_3 g_{\beta}(c_4 d(x,y),t)}{V(x,t^{1/\beta})},\qquad \forall (t,d(x,y))\in\Lambda_2. \tag{3.7}$$

Let $h(r):=\Psi(r)/r$. Then, (HK(Ψ)) is equivalent to

$$\frac{c_1}{V(x,\Psi^{-1}(t))}\exp\big(-\frac{c_2 d(x,y)}{h^{-1}(t/d(x,y))}\big)\le p_t(x,y)\le\frac{c_3}{V(x,\Psi^{-1}(t))}\exp\big(-\frac{c_4 d(x,y)}{h^{-1}(t/d(x,y))}\big), \tag{3.8}$$

$\forall x,y\in X$ and $t\in(0,\infty)$ where we let $d(x,y)/h^{-1}(t/d(x,y))=0$ if $d(x,y)=0$.

($LHK(\Psi)$): the first inequality of (3.8), ($UHK(\Psi)$): the second inequality of (3.8).

(VIII) $(\mathrm{VD})_{\mathrm{loc}}$: (VD) holds for $x \in X$, $0 < R \le 1$.

$(\mathrm{PI}(\bar\beta))_{\mathrm{loc}}$, $(\mathrm{EHI})_{\mathrm{loc}}$, $(\mathrm{CS}(\bar\beta))_{\mathrm{loc}}$, $(\mathrm{PHI}(\bar\beta))_{\mathrm{loc}}$ – define similarly.

$(\mathrm{HK}(\bar\beta))_{\mathrm{loc}}$: We require the bounds only for $t \in (0,1)$ – so only (3.6) is involved.

(IX) (a) We call φ a *cut-off function* for $A_1 \subset A_2$ if $\varphi = 1$ on A_1 and is zero on A_2^c.

(b) $(\mathrm{PI})_{\mathrm{loc}}$: $\forall c_1 > 0$, $\exists c_2 > 0$ s.t.

$$\int_B (f(x) - \overline{f}_B)^2 d\mu(x) \le c_2 \int_B d\Gamma(f,f)$$

for any ball $B = B(x, c_1) \subset X$ and $f \in \mathcal{F}$.

(c) $(\mathrm{CC})_{\mathrm{loc}}$: $\forall x_0 \in X$, $\exists$ a cut-off function $\varphi (= \varphi_{x_0})$ for $B(x_0, 1/2) \subset B(x_0, 1)$ s.t.

$$\int_{B(x_0,1)} d\Gamma(\varphi, \varphi) \le c_3 V(x_0, 1).$$

Remark. $(\mathrm{PI}(\bar\beta))_{\mathrm{loc}}$ for $\bar\beta \ge 2 \Rightarrow (\mathrm{PI})_{\mathrm{loc}}$, $\quad (\mathrm{CS}(\bar\beta))_{\mathrm{loc}}$ for $\bar\beta > 0 \Rightarrow (\mathrm{CC})_{\mathrm{loc}}$.

Weighted graphs with contr. weights $\Rightarrow (\mathrm{PI})_{\mathrm{loc}}, (\mathrm{CC})_{\mathrm{loc}}, (\mathrm{PI}(\bar\beta))_{\mathrm{loc}}, (\mathrm{CS}(\bar\beta))_{\mathrm{loc}}$ for $\bar\beta \ge 2$.

(X) (E(Ψ)): $\forall x_0 \in X$, $\forall R \geq 0$,

$$c_1 \Psi(R) \leq \mathbb{E}^{x_0}[\tau_{B(x_0,R)}] \leq c_2 \Psi(R), \tag{E(\Psi)}$$

where $\tau_A = \inf\{t \geq 0 : Y_t \notin A\}$.

$(\mathrm{E}(\Psi)_{\geq})$: the first inequality in (E(Ψ)), $(\mathrm{E}(\Psi)_{\leq})$: the second.

We summarize the conditions we have introduced:

(VD)	Volume doubling
(PI(Ψ))	Poincaré inequality
(EHI)	Elliptic Harnack inequality
(PHI(Ψ))	Parabolic Harnack inequality
(RES(Ψ))	Resistance exponent
(CS(Ψ))	Cut-off Sobolev inequality
(CC)	Controlled cut-off functions
(HK(Ψ))	Heat kernel estimates
(E(Ψ))	Walk dimension

When $\bar{\beta} = \beta$, we would write (...(β)) instead of (...(Ψ)), for instance (PI(β)) instead of (PI(Ψ)).

3.3 Main Theorems

Theorem 3.1 *X: MMD space or infinite con. weighted graph with contr. weights.*

The following are equivalent:

(a) X satisfies $(PHI(\Psi))$.

(b) X satisfies $(HK(\Psi))$.

(c) X satisfies (VD), $(PI(\Psi))$ and $(CS(\Psi))$.

(d) X satisfies (VD), (EHI) and $(RES(\Psi))$.

(e) X satisfies (VD), (EHI) and $(E(\Psi))$.

Stability We discuss two kinds of stability of (PHI(Ψ)).

Definition 3.2 *A property P is stable under bounded perturbation if whenever P holds for $(\mathcal{E}^{(1)},\mathcal{F})$, then it holds for $(\mathcal{E}^{(2)},\mathcal{F})$, provided*

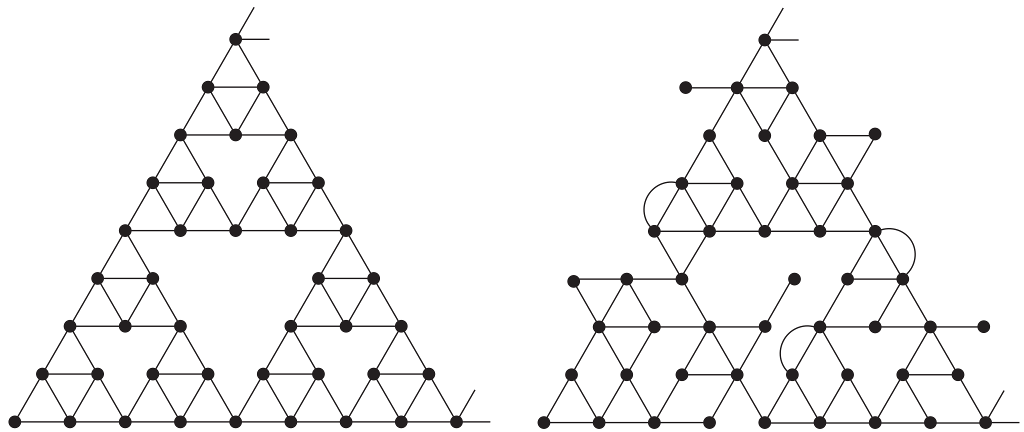
$$c_1\mathcal{E}^{(1)}(f,f) \le \mathcal{E}^{(2)}(f,f) \le c_2\mathcal{E}^{(1)}(f,f), \qquad \textit{for all } f \in \mathcal{F}. \tag{3.9}$$

Lemma 3.3 (Le Jan [64]) *X: MMD space. Suppose $(\mathcal{E}^{(i)},\mathcal{F}), i = 1,2$ are str. loc. reg. D-forms that satisfy (3.9). Then the energy measures $\Gamma^{(i)}$ satisfy*

$$c_1 d\Gamma^{(1)}(f,f) \le d\Gamma^{(2)}(f,f) \le c_2 d\Gamma^{(1)}(f,f), \qquad \textit{for all } f \in \mathcal{F}.$$

By this lemma, $PI(\Psi)$ and $CS(\Psi)$ are stable under bounded perturbations.

Theorem 3.4 *Let X be a MMD space. Then (PHI(Ψ)) and (HK(Ψ)) are stable under bounded perturbations.*



Rough isometry (M. Kanai in [52.53])

Definition 3.5 $(X_i, d_i, \mu_i), i = 1, 2$: *a MM space or a weighted graph.*

$\varphi : X_1 \to X_2$ *is a rough isometry if* $\exists c_1 > 0, c_2, c_3 > 1$ *s.t.*

$$X_2 = \bigcup_{x \in X_1} B_{d_2}(\varphi(x), c_1),$$

$$c_2^{-1}(d_1(x,y) - c_1) \le d_2(\varphi(x), \varphi(y)) \le c_2(d_1(x,y) + c_1),$$

$$c_3^{-1}\mu_1(B_{d_1}(x, c_1)) \le \mu_2(B_{d_2}(\varphi(x), c_1)) \le c_3\mu_1(B_{d_1}(x, c_1)).$$

If $\exists$ *a rough isometry between two spaces they are said to be roughly isometric.*

Stability of (PHI(Ψ)) under rough isometries.

Theorem 3.6 *X_i: a MMD space satisfying $(VD)_{\rm loc}$+$(PI)_{\rm loc}$ or a weighted graph with contr. weights. Suppose $\exists \varphi : X_1 \to X_2$ rough isom. Let $\Psi_i(s) = s^{\bar\beta_i} 1_{\{s\le 1\}} + s^{\beta} 1_{\{s\ge 1\}}$.*

(a) Suppose that X_2 satisfies $(PI(\bar\beta_2))_{\rm loc}$.

If X_1 satisfies (VD), $(CC)_{\rm loc}$ and $(PI(\Psi_1))$ then X_2 satisfies (VD) and $(PI(\Psi_2))$.

(b) Suppose that X_2 satisfies $(CS(\bar\beta_2))_{\rm loc}$.

If X_1 satisfies (VD) and $(CS(\Psi_1))$ then X_2 satisfies (VD) and $(CS(\Psi_2))$.

So, (PHI(Ψ)) is stable under rough isom., given suitable local reg. of the two spaces.

Examples 1) S.G. graphs in the last page satisfies $(PHI(\log 5/\log 2))$ for $R \ge 1$.

2) Fractal-like manifold in P 21: 2-dimensional Riemannian manifold

$\mathcal{L} = \sum_{i,j=1}^{2} \frac{\partial}{\partial x_i}(a_{ij}(x)\frac{\partial}{\partial x_j})$ on the manifold which satisfies the uniform elliptic condition enjoys $(HK(2))$ for $t \le 1 \vee d(x,y)$ and $(HK(\log 5/\log 2))$ for $t \ge 1 \vee d(x,y)$.

4 Proof of Theorem 3.1

Recall that $h(r) = \Psi(r)/r$. We give some inequalities.

$$p_t(x,y) \le \frac{C_1}{V(x,\Psi^{-1}(t))}, \qquad \forall x,y \in X, t>0. \qquad (DUHK(\Psi))$$

$$P^x(\tau_{B(x,r)} \le t) \le C_2 \exp\big(-\frac{C_3 r}{h^{-1}(t/r)}\big), \qquad \forall x \in X, r,t>0. \qquad (ELD(\Psi))$$

$$p_t(x,x) \ge \frac{C_4}{V(x,\Psi^{-1}(t))}, \qquad \forall x \in X, t>0. \qquad (DLHK(\Psi))$$

$$p_t(x,y) \ge \frac{C_5}{V(x,\Psi^{-1}(t))}, \qquad \forall x,y \in X, t>0 \ \text{ with } \ \Psi(d(x,y)) \le C_6 t. \quad (NLHK(\Psi))$$

4.1 Proof of $(e) \Rightarrow (b)$

For simplicity, we assume the existence of the (cont.) heat kernel and prove the following;

$$(\mathrm{VD}) + (DUHK(\Psi)) + (\mathrm{EHI}) + (E(\Psi)) \Rightarrow (HK(\Psi)).$$

STEP 1: PROOF OF $(E(\Psi)) \Rightarrow (ELD(\Psi))$.

Lemma 4.2 (Barlow-Bass) $\{\xi_i\}$*: non-negative random variables.*

Suppose $\exists 0 < p < 1, a > 0$ *s.t.*

$$P(\xi_i \le t | \sigma(\xi_1, \cdots, \xi_{i-1})) \le p + at, \qquad \forall t > 0.$$

$$\Rightarrow \qquad \log P(\sum_{i=1}^n \xi_i \le t) \le 2(\frac{ant}{p})^{1/2} - n \log \frac{1}{p}.$$

PROOF. Let η be a r.v. with distri. $P(\eta \le t) = (p + at) \wedge 1$. Then,

$$E(e^{-\lambda \xi_i} | \sigma(\xi_1, \cdots, \xi_{i-1})) \le Ee^{-\lambda \eta} = p + \int_0^{(1-p)/a} e^{-\lambda t} a dt \le p + a\lambda^{-1}.$$

So,
$$\begin{aligned} P(\sum_{i=1}^n \xi_i \le t) &= P(e^{-\lambda \sum_{i=1}^n \xi_i} \ge e^{-\lambda t}) \le e^{\lambda t} E e^{-\lambda \sum_{i=1}^n \xi_i} \\ &\le e^{\lambda t}(p + a\lambda^{-1})^n \le p^n \exp(\lambda t + \frac{an}{\lambda p}). \end{aligned}$$

The result follows on setting $\lambda = (an/(pt))^{1/2}$. □

Proof of $(E(\Psi)) \Rightarrow (ELD(\Psi))$. We first prove that $0 < \exists c_1 < 1, \exists c_2 > 0$ s.t.

$$P^x(\tau_{B(x,r)} \le s) \le 1 - c_1 + c_2 s/h(r) \qquad \text{for all } \; x \in X, \; s \ge 0. \tag{4.1}$$

Indeed, by the Markov property, for each $x \in X$ we have

$$E^x \tau_{B(x,r)} \le s + E^x[1_{\{\tau_{B(x,r)} > s\}} E^{X_s} \tau_{B(x,r)}] \le s + E^x[1_{\{\tau_{B(x,r)} > s\}} E^{X_s} \tau_{B(X_s,2r)}]. \tag{4.2}$$

Applying $(E(\Psi))$ and using the doubling property of h,

$$c_3 h(r) \le s + c_4 h(2r) P^x(\tau_{B(x,r)} > s) = s + c_5 h(r)(1 - P^x(\tau_{B(x,r)} \le s)). \tag{4.3}$$

Rearranging gives (4.1).

Next, let $l \ge 1$, $b = r/l$, and define stopping times $\sigma_i, \; i \ge 0$ by

$$\sigma_0 = 0, \quad \sigma_{i+1} = \inf\{t \ge \sigma_i : d(X_{\sigma_i}, X_t) \ge b\}.$$

Let $\xi_i := \sigma_i - \sigma_{i-1}$, $\mathcal{F}_t$: the filtration generated by $\{X_s : s \le t\}$, $\mathcal{G}_m := \mathcal{F}_{\sigma_m}$.

We have by (4.1)

$$P^x(\xi_{i+1} \leq t | \mathcal{G}_i) = P^{X_{\sigma_i}}(\tau_{B(X_{\sigma_i},b)} \leq t) \leq p + c_2 t/h(b),$$

where $0 < p < 1$. As $d(X_{\sigma_i}, X_{\sigma_{i+1}}) = b$, we have $d(X_0, X_{\sigma_l}) \leq r$, so that $\sigma_l = \sum_{i=1}^{l} \xi_i \leq \tau_{B(X_0,r)}$. So, by Lemma 4.2,

$$\log P^x(\tau_{B(x,r)} \leq t) \leq 2p^{-1/2}(\frac{c_2 lt}{h(r/l)})^{1/2} - l\log(1/p) = c_6(\frac{lt}{h(r/l)})^{1/2} - c_7 l.$$

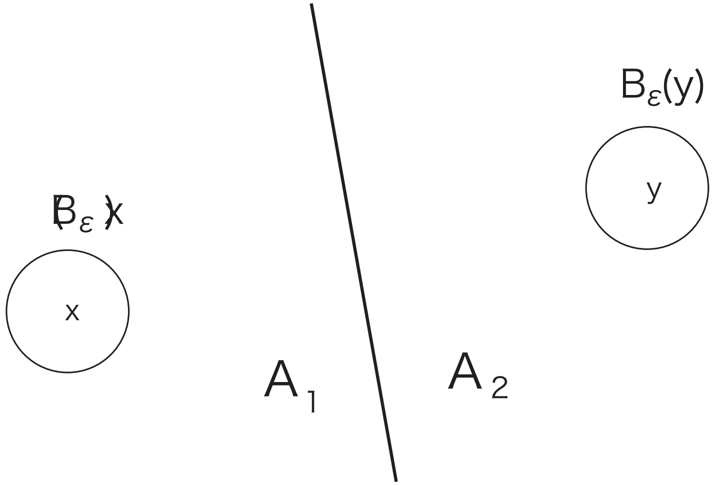
Now take $l_0 \in \mathbb{N}$ the largest integer l that satisfies

$$c_7 l/2 > c_6(\frac{lt}{h(r/l)})^{1/2}. \tag{4.4}$$

This is equivalent to $r/l > h^{-1}(c_8 t/r)$ where $c_8 = 4c_6^2/c_7^2$. Note: if $r \leq h^{-1}(c_8 t/r)$, then $(ELD(\Psi))$ holds by taking $c_1 > 0$ large. So assume (4.4) holds for small $l \in \mathbb{N}$. Then,

$$l_0 < \frac{r}{h^{-1}(c_8 t/r)} \leq l_0 + 1, \quad \text{and} \quad \log P^x(\tau_{B(x,r)} \leq t) \leq -c_7 l_0/2.$$

We thus obtain $(ELD(\Psi))$. □



<u>Step 2: Proof of $(VD)+(DUHK(\Psi))+(ELD(\Psi))\Rightarrow(UHK(\Psi))$.</u>

Fix $x\neq y$ and t and let $r:=d(x,y)$, $\epsilon<r/6$.

Let $\bar\mu_x=\mu|_{B_\epsilon(x)}$, $A_1=\{z\in X: d(z,x)\le d(z,y)\}$ and $A_2=X-A_1$. Then

$$\begin{aligned} P^{\bar\mu_x}(Y_t\in B_\epsilon(y)) &= P^{\bar\mu_x}(Y_t\in B_\epsilon(y),Y_{\frac{t}{2}}\in A_1)\\ &\quad +P^{\bar\mu_x}(Y_t\in B_\epsilon(y),Y_{\frac{t}{2}}\in A_2)\equiv I_1+I_2. \end{aligned}$$

Now, $\quad I_2 \le P^{\bar\mu_x}(Y_t\in B_\epsilon(y),\tau<\dfrac{t}{2}) = E^{\bar\mu_x}(1_{\tau<t/2}\displaystyle\int_{B_\epsilon(y)} p_{t-\tau}(Y_\tau,w)d\mu(w))$

$$\leq P^{\bar{\mu}_x}(\tau < t/2) \sup_{z \in B(x,r/2) \cup B_\epsilon(y)} p_{t/2}(z,z)\mu(B_\epsilon(y)), \qquad \text{where } \tau := \tau_{B(x,r/2)}.$$

By $(ELD(\Psi))$, we obtain

$$I_2 \leq c_1 \big(\sup_{z \in B(x,r/2) \cup B_\epsilon(y)} p_{t/2}(z,z) \big) \mu(B_\epsilon(x)) \mu(B_\epsilon(y)) \exp\big(-\frac{c_2 r}{h^{-1}(t/r)} \big).$$

For I_1, by the symmetry of $p_t(x,y)$,

$$P^{\bar{\mu}_x}(Y_t \in B_\epsilon(y), Y_{\frac{t}{2}} \in A_1) = P^{\bar{\mu}_y}(Y_t \in B_\epsilon(x), Y_{\frac{t}{2}} \in A_1)$$

which is bounded in exactly the same way as I_2,where x and y are changed. So,

$$P^{\bar{\mu}_x}(Y_t \in B_\epsilon(y)) \leq c_1 \big(\sup_{z \in B(x,r/2) \cup B(y,r/2)} p_{t/2}(z,z) \big) \mu(B_\epsilon(x)) \mu(B_\epsilon(y)) \exp\big(-\frac{c_2 r}{h^{-1}(t/r)} \big).$$

By $(DUHK(\Psi))$ and (VD),

$$\sup_{z \in B(x,r/2) \cup B(y,r/2)} p_{t/2}(z,z) \leq \frac{c_3}{V(x, \Psi^{-1}(t))} \big(\frac{r + \Psi^{-1}(t)}{\Psi^{-1}(t)} \big)^\alpha.$$

If $\Psi(r) \leq t$, this is bounded by $c_4 V(x, \Psi^{-1}(t))^{-1}$. If $\Psi(r) > t$, then, $\forall \epsilon > 0$, $\exists c_\epsilon > 0$ s.t.

$$\big(\frac{r + \Psi^{-1}(t)}{\Psi^{-1}(t)} \big)^\alpha \exp\big(-\frac{\epsilon r}{h^{-1}(t/r)} \big) \leq c_\epsilon.$$

This is because $M = r/\Psi^{-1}(t) \Leftrightarrow h(r/M) = tM/r \Rightarrow M < r/h^{-1}(t/r)$. In any case,

$$P^{\bar{\mu}_x}(Y_t \in B_\epsilon(y)) \leq \frac{c_5}{V(x, \Psi^{-1}(t))} \mu(B_\epsilon(x)) \mu(B_\epsilon(y)) \exp\big(-\frac{c_6 r}{h^{-1}(t/r)}\big).$$

Dividing both sides by $\mu(B_\epsilon(x))$, $\mu(B_\epsilon(y))$ and using cont. of $p_t(x, y)$ gives $(UHK(\Psi))$.□

STEP 3: PROOF OF $(VD) + (ELD(\Psi)) \Rightarrow (DLHK(\Psi))$. Using (4.1),

$$P^x(Y_t \notin B(x, r)) \leq P(\tau_{B(x,r)} \leq t) \leq c_1 \exp\big(-\frac{c_2 r}{h^{-1}(t/r)}\big).$$

Hence, by choosing r s.t. $c_3\Psi(r) < t < c_4\Psi(r)$ for $\exists c_3, c_4 > 0$, we have

$$P^x(Y_t \notin B(x, r)) \leq c_5 < 1.$$

Thus $P^x(Y_t \in B(x, r)) \geq 1 - c_5 > 0$. By Cauchy-Schwarz,

$$(1 - c_5)^2 \leq P^x(Y_t \in B(x, r))^2 = \Big(\int_{B(x,r)} p_t(x, z) d\mu(z)\Big)^2 \leq V(x, r) p_{2t}(x, x).$$

Now, using the lower bound of our choice of t and (VD), we obtain the result. □

STEP 4: PROOF OF (VD) + $(DUHK(\Psi))$ + (EHI) + (E(Ψ)) $\Rightarrow$ $(NLHK(\Psi))$.

(Scketch) Fix $x \in X$, $t > 0$ and set $R := \Psi^{-1}(t/\varepsilon)$ ($\varepsilon > 0$ will be chosen later).

• Similarly to Step 3, if $\varepsilon > c_2$, we obtain

$$p_t^B(x,x) \geq \frac{c_1}{V(x,\Psi^{-1}(t))}, \qquad \text{where } \ B := B(x,R). \tag{4.6}$$

• Set $f(y) := \partial_t p_t^B(x,y)$. Applying Proposition 9.9 (time derivative) to p_t^B,

$$|f(y)| \leq \frac{2}{t}\sqrt{p_{t/2}^B(x,x)p_{t/2}^B(y,y)} \leq \frac{2}{t}\sqrt{p_{t/2}(x,x)p_{t/2}(y,y)}, \qquad y \in B.$$

By $(DUHK(\Psi))$ and (VD), $\exists \alpha, \alpha' > 0$ s.t.

$$p_{t/2}(y,y) \leq \frac{c_1}{V(y,\Psi^{-1}(t))} \leq \frac{c_1}{V(x,\Psi^{-1}(t))}(1+\frac{d(x,y)}{\Psi^{-1}(t)})^\alpha \leq \frac{c_1(1+\varepsilon^{-\alpha'})^\alpha}{V(x,\Psi^{-1}(t))}, \qquad \forall y \in B.$$

Hence, by (VD), we have

$$|f(y)| \leq \frac{c_2(1+\varepsilon^{-\alpha'})^{\alpha/2}}{tV(x,\Psi^{-1}(t))}, \qquad \forall y \in B. \tag{4.7}$$

- Define $u(y) = p_t^B(x,y)$. Then, $\partial_t u = \Delta_B u$, so $u = -G^B(\partial_t u) = G^B f$, where $G^B = (-\Delta_B)^{-1}$ is the Green operator. Let $\gamma > \alpha\alpha'/2$ and apply Proposition 9.6 (Oscillation inequality, (EHI) is used here) with $\varepsilon^{\gamma+1}$ instead of ε. Then, $\exists\delta > 0$ s.t. $0 < \forall r < R$,

$$\mathrm{Osc}_{B(x,\delta r)} u \le 2(\bar{E}(x,r) + \varepsilon^{\gamma+1}\bar{E}(x,R)\|f\|_\infty,$$

where $\bar{E}(x,r) := \sup_z E^z[\tau_{B(x,r)}]$. By (E($\Psi$)) and (4.7), we obtain

$$\mathrm{Osc}_{B(x,\delta r)} u \le \frac{\Psi(r) + \varepsilon^{\gamma+1}\Psi(R)}{t} \cdot \frac{c_4(1+\varepsilon^{-\alpha'})^{\alpha/2}}{V(x,\Psi^{-1}(t))}.$$

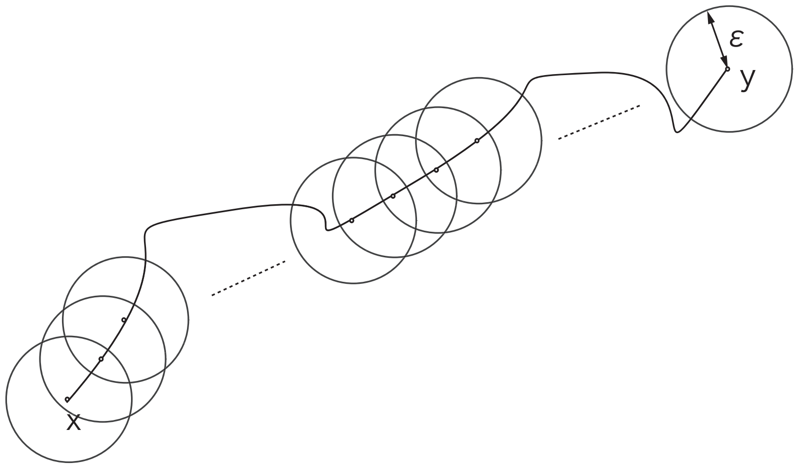
- By definition of R, $\frac{\varepsilon^{\gamma+1}\Psi(R)}{t} = \varepsilon^\gamma$.

Choose r by the eq. $\Psi(r) = \varepsilon^{\gamma+1}\Psi(R)$, (so $r \ge \delta' R$ for $\exists\delta' > 0$). Hence,

$$\mathrm{Osc}_{y\in B(x,\delta\delta' R)} p_t^B(x,y) \le \mathrm{Osc}_{B(x,\delta r)} u \le \frac{2c_4\varepsilon^\gamma(1+\varepsilon^{-\alpha'})^{\alpha/2}}{V(x,\Psi^{-1}(t))} \to 0 \ \ (\text{as } \varepsilon \to 0). \tag{4.8}$$

Choosing ε small enough and combining (4.8) with (4.6), we conclude that

$$p_t(x,y) \ge p_t^B(x,y) \ge \frac{c_1/2}{V(x,\Psi^{-1}(t))}, \qquad \forall y \in B(x,\delta\delta' R). \ \square$$



<u>STEP 5: PROOF OF (VD) + $(NLHK(\Psi)) \Rightarrow (LHK(\Psi))$.</u> Chain argument.

(Scketch) Let $\varepsilon = \varepsilon(t, d(x,y)) > 0$ be s.t.

$$c_1 t \leq h(\varepsilon) d(x,y) \leq c_2 t. \tag{4.9}$$

Due to $(NLHK(\Psi))$, we should only consider the case $\Psi(d(x,y)) > C_6 t$, which means $\varepsilon < c_3 d(x,y)$ for $\exists c_3 > 0$. Take $N \in \mathbb{N}$ s.t. $N \asymp d(x,y)/\varepsilon$.

Let $\{x_i\}_{i=0}^N$ be such that $x_0 = x, x_N = y$ and $d(x_i, x_{i+1}) \leq \varepsilon$ for $i = 0, 1, \cdots, N-1$.

(Such a seq. exists by the choice of N and by the fact that d is a geodesic met.) Then,

$$\begin{aligned} p_t(x,y) &= \int_X \cdots \int_X p_{t/N}(x,z_1)p_{t/N}(z_1,z_2)\cdots p_{t/N}(z_{N-1},y)d\mu(z_1)\cdots d\mu(z_{N-1}) \\ &\geq \int_{B(x_1,\varepsilon)} \cdots \int_{B(x_{N-1},\varepsilon))} p_{t/N}(x,z_1)\cdots p_{t/N}(z_{N-1},y)d\mu(z_1)\cdots d\mu(z_{N-1}). \end{aligned}$$

Clearly $d(z_i,z_{i+1}) \leq 3\varepsilon$. Now, by the choice of ε and N, we have $\varepsilon \asymp \Psi^{-1}(\frac{t}{N})$.

This together with $(NLHK(\Psi))$ and (VD) and (4.12), we have

$$p_{t/N}(z_i,z_{i+1}) \geq \frac{c_6}{V(z_i,\Psi^{-1}(t/N))} \geq \frac{c_7}{V(x_i,\Psi^{-1}(t/N))} \geq \frac{c_8}{V(x_i,\varepsilon)}.$$

$$\text{So,}\ \ p_t(x,y) \geq \frac{c_8}{V(x,\Psi^{-1}(t/N))}\prod_{i=1}^{N-1}\frac{c_8\cdot V(x_i,\varepsilon)}{V(x_i,\varepsilon)} \geq \frac{c_8^{-N}}{V(x,\Psi^{-1}(t/N))} \geq \frac{\exp(-c_9N)}{V(x,\Psi^{-1}(t))}.$$

On the other hand, by (4.9) we have $h^{-1}(t/d(x,y)) \leq c_{11}\varepsilon$, so that

$$N \asymp \frac{d(x,y)}{\varepsilon} \leq c_{11}\frac{d(x,y)}{h^{-1}(t/d(x,y))}.$$

We thus obtain $(LHK(\Psi))$. □

4.2 Proof of $(c) \Rightarrow (d)$

Lemma 4.4

$$(VD) + (PI(\Psi)) + (CS(\Psi)) \Rightarrow (RES(\Psi)).$$

PROOF. $\underline{\text{(VD)} + \text{(PI}(\Psi)) \Rightarrow \text{(RES}(\Psi))_{\geq}}$

f: attains the minimum in the variational formula of $R(B(x_0, R), B(x_0, 2R)^c)$.

$\overline{f} := \int_{B(x_0,3R)} f d\mu / V(x_0, 3R)$. Choose y_0 s.t. $d(x_0, y_0) = 5R/2$.

By (9.1) (due to (VD)), $V(y_0, R/2) \geq c_2 V(x_0, R)$.

Depending on $\overline{f} \geq 1/2$ or $\overline{f} < 1/2$, $|f - \overline{f}| \geq 1/2$ on either $B(x_0, R)$ or $B(y_0, R/2)$,

and then using (PI(Ψ)) we have

$$\begin{aligned} V(x_0, R) &\leq c_3 \int_{B(x_0,3R)} (f - \overline{f})^2 d\mu \leq c_4 \Psi(R) \int_{B(x_0,3R)} d\Gamma(f, f) \\ &= c_4 \Psi(R) R(B(x_0, R), B(x_0, 2R)^c)^{-1}. \qquad \square \end{aligned}$$

(VD) + (CS(Ψ)) $\Rightarrow$ (RES(Ψ))$_\leq$

φ: a cut-off function for $B(x_0, R)$ given by (CS(Ψ)).

Taking $f \equiv 1$, $I = B(x_0, R)$ and $I^* = B(x_0, 2R)$ in (3.5), we obtain

$$R(B(x_0, R/2), B(x_0, R)^c)^{-1} \leq \int_I d\Gamma(\varphi, \varphi) \leq c_6 \Psi(R)^{-1} \int_{I^*} d\mu \leq c_7 \frac{V(x_0, R)}{\Psi(R)}. \qquad \square$$

The rest is to show (VD) + (PI(Ψ)) + (CS(Ψ)) $\Rightarrow$ (EHI).

Recall the Moser's argument in subsection 2.4. The crucial loss for the case $\beta \neq 2$ is in using the bound (2.6); one needs a cutoff function φ such that the final term in (2.7) can be controlled by a term of order $R^{-\beta}$.

Fix $x \in X$, $R > 0$. $\varphi = \varphi_{x,R}$: the cut-off function in (CS(Ψ)).

Define the measure $\gamma = \gamma_{x,R}$ by

$$d\gamma = d\mu + \Psi(R) d\Gamma(\varphi, \varphi).$$

The first step in the argument is to use (CS(Ψ)) to obtain a weighted Sobolev inequality.

Proposition 4.5 *Let $s \le R$ and $J \subset B(x_0, R)$ be a finite union of balls of radius s. $\exists \kappa > 1, c_1 > 0$ s.t.*

$$(\mu(J)^{-1}\int_J |f|^{2\kappa} d\gamma)^{1/\kappa} \le c_1(\Psi(R)\mu(J)^{-1}\int_{J^s} d\Gamma(f,f) + (s/R)^{-2\theta}\mu(J)^{-1}\int_J f^2 d\gamma),$$

where $J^s = \{y : d(y, J) \le s\}$.

(Strategy of the proof): Prove weighted Poincaré ineq. first, and then prove the weighted Nash ineq., which deduce the desired inequality. See subsection 9.8 for details.

The next result is the generalization of Lemma 4 of [69] to the case of a MMD space.

Lemma 4.6 *Let D be a domain in X, let u be positive and harmonic in D, $v = u^k$, where $k \in \mathbb{R}$, $k \ne \frac{1}{2}$, and let η be supported in D. Suppose $\int_D d\Gamma(\eta, \eta) < \infty$, then*

$$\int_D \eta^2 d\Gamma(v, v) \le (\frac{2k}{2k-1})^2 \int_D v^2 d\Gamma(\eta, \eta).$$

u: harmonic and nonnegative in $B(x_0, 4R)$. (W.l.o.g. suppose u is strictly positive.)

Remark. We do not initially have any *a priori* continuity for u.

Proposition 4.7 *Let v be either u or u^{-1}.*

$\exists c_1$ s.t. if $B(x, 2r) \subset B(x_0, 4R)$ and $0 < q < 2$, then

$$ess\ sup_{B(x,r/2)} v^{2q} \leq c_1 V(x, 2r)^{-1} \int_{B(x,2r)} (\Psi(r) d\Gamma(v^q, v^q) + v^{2q} d\mu).$$

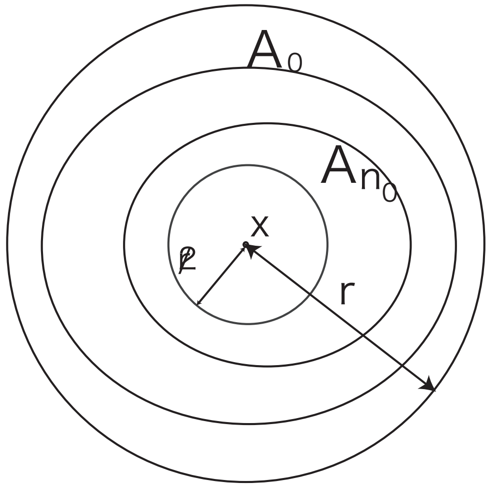
Proof. (Sketch) φ_0: cut-off function given by (CS(Ψ)) for $B(x, r)$. $h_n := 1 - 2^{-n}$, and

$$\varphi_k(x) := (\varphi_0(x) - h_k)^+, \quad d\gamma_0 := d\mu + \Psi(r) d\Gamma(\varphi_0, \varphi_0), \quad A_k := \{x : \varphi_0(x) > h_k\}.$$

Then, $\mu(A_k) \asymp V(x, r) =: V$.

[Hölder cond. on φ_0 by (CS(Ψ))] $\Rightarrow$ [if $x \in A_{k+1}, y \in A_k^c$, then $d(x, y) \geq c_3 r 2^{-k/\theta}$] $\Rightarrow$ [$\varphi_k > c_4 2^{-k}$ on $A_{k+1}^{s_k} =: A'_{k+1}$ where $s_k = \frac{1}{2} c_3 r 2^{-k/\theta}$]. By Proposition 4.5 with $f = v^p$,

$$(V^{-1} \int_{A_{k+1}} f^{2\kappa} d\gamma_0)^{1/\kappa} \leq c_6 V^{-1} [\Psi(r) \int_{A'_{k+1}} d\Gamma(f, f) + 2^{2k} \int_{A_k} f^2 d\gamma_0].$$



By Lemma 4.6, we have the 'converse to the Poincaré inequality' for $f = v^p$;

$$\begin{aligned}\Psi(r)\int_{A'_{k+1}} d\Gamma(f,f) &\le \Psi(r)(c_7 2^{-k})^{-2}\int_{A'_{k+1}}\varphi_k^2 d\Gamma(f,f) \le c_8 2^{2k}\Psi(r)\int_{A_k}\varphi_k^2 d\Gamma(f,f)\\ &\le c_9 2^{2k}\Psi(r)(\frac{2p}{2p-1})^2\int_{A_k} f^2 d\Gamma(\varphi_k,\varphi_k) \le c_{10}2^{2k}(\frac{2p}{2p-1})^2\int_{A_k} f^2 d\gamma_0.\end{aligned}$$

So,
$$(V^{-1}\int_{A_{k+1}} f^{2\kappa}d\gamma_0)^{1/\kappa} \le c_{11}(\frac{2p}{2p-1})^2 2^{2k}V^{-1}\int_{A_k} f^2 d\gamma_0. \tag{4.21}$$

Now, argument similar to the first part of Moser's argument.

$q_0 := q'\kappa^{-i}$ for $\exists i$, $p_n := 2q_0\kappa^n$, and $\Psi_k =[\mu(A_k)^{-1}\int_{A_k} v^{p_k}d\gamma_0]^{1/p_k}$.

Note that $p_{k+1}/2\kappa = p_k/2$. Applying (4.21) to $f = v^{p_{k+1}/(2\kappa)} = v^{p_k/2}$ we have

$$\Psi_{k+1}^{p_{k+1}/\kappa} = (\mu(A_{k+1})^{-1} \int_{A_{k+1}} v^{p_{k+1}} d\gamma_0)^{1/\kappa} \leq c_{13} 2^{2k} (\mu(A_k))^{-1} \int_{A_k} v^{p_k} d\gamma_0) = c_{13} 2^{2k} \Psi_k^{p_k}.$$

$$\text{Hence,} \qquad \log \Psi_m \leq \log \Psi_0 + \sum_{k=1}^{m} p_k^{-1} \log(c_{13} 2^{2k}). \tag{4.22}$$

As the sum in (4.22) converges and $\text{ess sup}_{B(x,r/2)} v \leq \limsup_{m\to\infty} \Psi_m$,

$$\text{ess sup}_{B(x,r/2)} v \leq c_{14} \Psi_0 \leq c_{15} (V^{-1} \int_{B(x,r)} v^{2q_0} d\gamma_0)^{1/(2q_0)}.$$

Let $q \in (0,2)$; we can take $q_0 = q' \kappa^{-i} < q$. By the weighted Poincaré ineq. (Prop 9.20),

$$(\int_{B(x,r)} \frac{v^{2q_0}}{V} d\gamma_0)^{q/q_0} \leq c_{16} \int_{B(x,r)} \frac{v^{2q}}{V} d\gamma_0 \leq c_{18} V^{-1} \int_{B(x,2r)} (\Psi(r) d\Gamma(v^q, v^q) + v^{2q} d\mu).$$

So, we conclude

$$\text{ess sup}_{B(x,r/2)} v^{2q} \leq c_{18} V(x,2r)^{-1} \int_{B(x,2r)} (\Psi(r) d\Gamma(v^q, v^q) + v^{2q} d\mu). \quad \square$$

Recall that φ is a cut-off function for $B(x_0, R)$ given by (CS(Ψ)). We define

$$Q(t) = \{x : \varphi(x) > t\}, \quad 0 < t < 1.$$

Corollary 4.8 *Let* $1 > s > t > 0$. *There exists* $\zeta > 2$ *such that if* $0 < q < \frac{1}{3}$,

$$ess\ sup_{Q(s)} v^{2q} \leq c_1 (s-t)^{-\zeta} V(x_0, R)^{-1} \int_{Q(t)} v^{2q} d\gamma.$$

The following corresponds to the second part of Moser's arguments.

Proposition 4.9 *Let* $w = \log u$, *and write* $\overline{w} = V(x_0, R)^{-1} \int_{B(x_0,R)} w \, d\mu$.

(a) $$\int_{B(x_0,2R)} d\Gamma(w,w) \leq c_1 \frac{V(x_0,R)}{\Psi(R)}.$$

(b) $$\int_{\{|w-\overline{w}|>A\} \cap Q(s)} d\gamma \leq c_2 \frac{V(x_0,R)}{A^2}, \qquad \textit{for} \quad 0 < t < s \leq 1.$$

To get the Harnack inequality.

- [68]: generalization of the John-Nirenberg inequality with a complicated proof.
- Bombieri [22]: avoid such an argument for elliptic second order diff. eqs.
- Moser ([67], Lemma 3) carried the idea over to the parabolic case
- Bombieri-Giusti ([23], Theorem 4): ineq. in an abstract setting ([72], Lemma 2.2.6)

Using these, we can show that Corollary 4.8 and Proposition 4.9 (b) give

$$\text{ess sup}_{B(x_0,R/2)} \log u \le c_1. \tag{4.26}$$

Let $v = u^{-1}$. The same argument implies

ess $\sup_{B(x_0,R/2)} \log v \le c_1$, or ess $\inf_{B(x_0,R/2)} \log u \ge -c_1$. Combining we deduce

$$e^{-c_1} \le \text{ess inf}_{B(x_0,R/2)} u \le \text{ess sup}_{B(x_0,R/2)} u \le e^{c_1}.$$

Theorem 4.10 *$\exists c_1$ s.t. if u is nonneg. and harmonic in $B(x_0, 4R)$, then*

$$\text{ess sup}_{B(x_0,R/2)} u \le c_1 \text{ess inf}_{B(x_0,R/2)} u.$$