Harnack Inequalities - An Introduction

Moritz Kassmann

Institut für Angewandte Mathematik
Sonderforschungsbereich 611
Universität Bonn
www.iam.uni-bonn.de/∼kassmann/

kassmann@iam.uni-bonn.de

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Carl-Gustav Axel von Harnack

- 1851 (Dorpat) - 1888 (Dresden).
- Father and brother theologians. Brother Carl-Gustav Adolf later founder of the Kaiser-Wilhem Gesellschaft (Max Planck Society).
- Student of Felix Klein in Erlangen, PhD in 1875.
- Professor at Darmstadt, Leipzig, Dresden.
- Well-known for a textbook and his translation of the book by Serret.
- 1887 - book on potential theory in two dimensions.

Axel von Harnack
(1851-1888)

Harnack's result

Theorem

Let $u : B_R(x_0) \subset \mathbb{R}^d \to \mathbb{R}$ be a harmonic function which is either non-negative or non-positive. Then the value of u at any point in $B_r(x_0)$ is bounded from above and below by the quantities

$$u(x_0)\Big(\frac{R}{R+r}\Big)^{d-2}\frac{R-r}{R+r} \quad \textit{and} \quad u(x_0)\Big(\frac{R}{R-r}\Big)^{d-2}\frac{R+r}{R-r}\,. \tag{1}$$

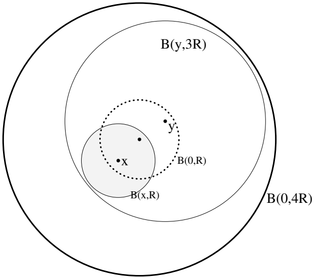
1. If $u : \mathbb{R}^d \to \mathbb{R}$ is harmonic and bounded from below or bounded from above then it is constant. (Liouville Theorem).
2. If $u : \{x \in \mathbb{R}^3; 0 < |x| < R\} \to \mathbb{R}$ is harmonic and satisfies $u(x) = o(|x|^{2-d})$ for $|x| \to 0$ then $u(0)$ can be defined in such a way that $u : B_R(0) \to \mathbb{R}$ is harmonic. (Removable Singularity Theorem).
3. Let $\Omega \subset \mathbb{R}^d$ be a domain and (u_n) be a sequence of monotonically increasing harmonic functions $u_n : \Omega \to \mathbb{R}$. Assume that there is $x_0 \in \Omega$ with $|u_n(x_0)| \leq K$ for all n. Then u_n converges uniformly on each subdomain $\Omega' \Subset \Omega$ to a harmonic function u. (Harnack's second convergence theorem).

„Modern“ Reformulation

Corollary

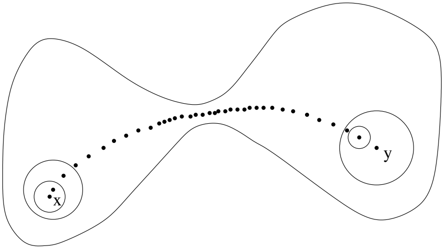
For any given domain $\Omega \subset \mathbb{R}^d$ *and subdomain* $\Omega' \Subset \Omega$ *there is a constant* $C = C(d, \Omega', \Omega) > 0$ *such that for any non-negative harmonic function* $u : \Omega \longrightarrow \mathbb{R}$

$$\sup_{x \in \Omega'} u(x) \leq C \inf_{x \in \Omega'} u(x) . \tag{2}$$



$\Omega = B(0, 4R)$; $\Omega' = B(0, R)$

$x, y \in B(0, R) \Rightarrow x \in B(y, 3R)$.



Comments on Early History

- [Poincaré, 1890] relies on Harnack inequalities and especially on convergence theorems.
- [Lichtenstein, 1912] proves a Harnack inequality for elliptic operators with differentiable coefficients including lower order terms in two dimensions.
- [Feller, 1930] extends this to any space dimension $d \in \mathbb{N}$.
- [Serrin, 1955] reduces the assumptions on the coefficients substantially and provides in two dimensions a Harnack inequality in the case where the leading coefficients are merely bounded; see also [Bers/Nirenberg 1955] for this result.
- [Cordes, 1956] relaxes the assumptions of [Nirenberg, 1953] for Hölder regularity.

Earl accounts in textbooks include [Lichtenstein, 1918] and [Kellogg, 1929].

A non-local version

Theorem

Let $\alpha \in (0,2)$ *and* $C(d,\alpha) = \frac{\alpha\Gamma(\frac{d+\alpha}{2})}{2^{1-\alpha}\pi^{\frac{d}{2}}\Gamma(1-\frac{\alpha}{2})}$. *Let* $u : \mathbb{R}^d \to \mathbb{R}$ *be a* **non-negative function** *satisfying*

$$-(-\boldsymbol{\Delta})^{\alpha/2}\mathbf{u}(\mathbf{x}) = C(d,\alpha) \lim_{\varepsilon \to 0} \int\limits_{|h|>\varepsilon} \frac{u(x+h)-u(x)}{|h|^{d+\alpha}}\, dh = 0 \quad \forall\, x \in B_R(0)\,.$$

Then for any $y, y' \in B_R(0)$

$$u(y) \leq \left| \frac{R^2 - |y|^2}{R^2 - |y'|^2} \right|^{\alpha/2} \left| \frac{R - |y|}{R + |y'|} \right|^{-d} u(y')\,. \tag{3}$$

Look at $\alpha \to 2$!!!

Proof: Use Poisson kernel estimates of [M. Riesz, 1938]. See chapter IV, paragraph 5 in [Landkof, 1972].

Local versus non-local behavior

Theorem (Kassmann)

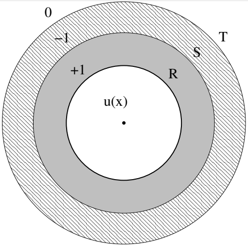
Let $R > 0$. There exists a function $u : \mathbb{R}^d \to \mathbb{R}$ satisfying

$$|u(x)| \leq 1 \forall x \in \mathbb{R}^d \qquad \textit{and } (-\boldsymbol{\Delta})^{\alpha/2}\mathbf{u}(\mathbf{x}) = 0, u(x) \geq 0 \ \forall x \in B_R(0)\,,$$

and at the same time $u(0) = 0$. Therefore the classical Harnack inequality fails for the operator $\mathcal{L}$.

Let $g : \mathbb{R}^2 \setminus B_R(0) \to \mathbb{R}$ be:

$$g(x) = \begin{cases} 1 & ; R \leq |x| < S\,, \\ -1 & ; S \leq |x| < T\,, \\ 0 & ; T \leq |x|\,. \end{cases}$$



$$u(y) := C(d,\alpha)\big(R^2 - |y|^2\big)^{\frac{\alpha}{2}} \int\limits_{\mathbb{R}^2 \setminus B_R(0)} g(x)\big(R^2 - |x|^2\big)^{\frac{-\alpha}{2}} |x-y|^{-2}\, dx\,,$$

Equations in Divergence Form - Setup

Assume $\Omega \subset \mathbb{R}^d$ to be a bounded domain and $x \mapsto A(x) = \big(a_{ij}(x)\big)_{i,j=1,\dots,d}$ satisfies $a_{ij} \in L^\infty(\Omega)$ $(i,j = 1,\dots,d)$ and

$$\lambda|\xi|^2 \leq a_{ij}(x)\xi_i\xi_j \leq \lambda^{-1}|\xi|^2 \quad \forall x \in \Omega, \xi \in \mathbb{R}^d \tag{4}$$

for some $\lambda > 0$. $u \in H^1(\Omega)$ is called **subsolution** of the uniformly elliptic equation

$$-\operatorname{div}\big(A(.)\nabla u\big) = -D_i\big(a_{ij}(.)D_j u\big) = f \in L^q(\Omega), q > d/2\,. \tag{5}$$

in Ω if

$$\int_\Omega a_{ij} D_i u D_j \phi \leq \int_\Omega f\phi \qquad \text{for any } \phi \in H^1_0(\Omega), \phi \geq 0 \text{ in } \Omega\,. \tag{6}$$

Here, $H^1(\Omega)$ denotes the Sobolev space of all $L^2(\Omega)$ functions with generalized first derivatives in $L^2(\Omega)$. The notion of **supersolution** is analogous. A function $u \in H^1(\Omega)$ satisfying $\int_\Omega a_{ij} D_i u D_j \phi = \int_\Omega f\phi$ for any $\phi \in H^1_0(\Omega)$ is called a **weak solution** in Ω.

Elliptic Equations in Divergence Form - Local Regularity 1

Local Boundedness: For any non-negative subsolution $u \in H^1(\Omega)$ of (5) and any $B_R(x_0) \Subset \Omega$, $0 < r < R$, $p > 0$ with $c = c(d, \lambda, p, q) > 0$

$$\sup_{B_r(x_0)} u \le c\Big\{(R-r)^{-d/p}\|u\|_{L^p(B_R(x_0))} + R^{2-\frac{d}{q}}\|f\|_{L^q(B_R(x_0))}\Big\}\,, \tag{7}$$

Weak Harnack Inequality: For any non-negative supersolution $u \in H^1(\Omega)$ of (5) and any $B_R(x_0) \Subset \Omega$, $0 < \theta < \rho < 1$, $0 < p < \frac{n}{n-2}$ with $c = c(d, \lambda, p, q, \theta, \rho) > 0$

$$\inf_{B_{\theta R(x_0)}} u + R^{2-\frac{d}{q}}\|f\|_{L^q(B_R(x_0))} \ge c\Big\{R^{-d/p}\|u\|_{L^p(B_{\rho R}(x_0))}\Big\}\,. \tag{8}$$

Harnack Inequality: For any non-negative weak solution $u \in H^1(\Omega)$ of (5) and any $B_R(x_0) \Subset \Omega$ with $c = c(d, \lambda, q) > 0$

$$\sup_{B_{R/2}(x_0)} u \le c\Big\{\inf_{B_{R/2}(x_0)} u + R^{2-\frac{d}{q}}\|f\|_{L^q(B_R(x_0))}\Big\}\,. \tag{9}$$

Elliptic Equations in Divergence Form - Local Regularity 2

Corollary (DeGiorgi, Nash, Moser)

Let $f \in L^q(\Omega)$, $q > d/2$. There exist two constants $\alpha = \alpha(d, q, \lambda) \in (0, 1)$, $c = c(d, q, \lambda) > 0$ such that for any weak solution $u \in H^1(\Omega)$ of (5) $u \in C^\alpha(\Omega)$. In addition, for any $B_R \Subset \Omega$ with some $c = c(d, \lambda, q) > 0$:

$$|u(x) - u(y)| \leq cR^{-\alpha}|x - y|^\alpha \Big\{ R^{-d/2}\|u\|_{L^2(B_R)} + R^{2-\frac{d}{q}}\|f\|_{L^q(B_R)} \Big\} \; \forall\, x, y \in B_{R/2}.$$

Though, the method of reduction of oscillations goes back to Harnack [1887]:

> *„Let u be a harmonic function on a ball with radius r. Denote by D the oscillation of u on the boundary of the ball. Then the oscillation of u on a inner ball with radius $\rho < r$ is not greater than $\frac{4}{\pi}\arcsin(\frac{\rho}{r})D$.“*

Interesting: Already Harnack uses the auxiliary function $v(x) = u(x) - \frac{M+m}{2}$ where M denotes the maximum of u and m the minimum over a ball.

Nonlinear Elliptic PDE in Divergence Form - 1

As turns out, the concept of Harnack inequalities is universal. It does not depend on the linear structure of the differential operators.

[Serrin, 1964] and [Trudinger, 1967] extend Moser's results to the situation of nonlinear elliptic equations of the following type:

$$\operatorname{div} \mathbf{A}(.,u,\nabla u) + B(.,u,\nabla u) = 0 \quad \text{weakly in } \Omega, \quad u \in W^{1,p}_{\mathrm{loc}}(\Omega),\ p > 1\,.$$

with

$$|\mathbf{A}(x,z,\mathbf{v})| \leq c_0 |\mathbf{v}|^{p-1} + c_1(x)|z|^{p-1}\,, \tag{10}$$

$$\mathbf{v} \cdot \mathbf{A}(x,z,\mathbf{v}) \geq |\mathbf{v}|^p - c_2(x)|z|^p\,, \tag{11}$$

$$|B(x,z,\mathbf{v})| \leq d_0 |\mathbf{v}|^p + d_1(x)|\mathbf{v}|^{p-1} + d_2(x)|z|^{p-1}\,, \tag{12}$$

for $(x,z,\mathbf{v}) \in \Omega \times \mathbb{R} \times \mathbb{R}^d$. Here $p > 1$, $c_0, d_0 > 0$ and

$$|c_1|^{1/p-1}, |c_2|^{1/p}, d_1, |d_2|^{1/p} \in L^q(\Omega), \quad q > d \text{ if } p \geq d \text{ and } q = d \text{ if } p < d\,.$$

Nonlinear Elliptic PDE in Divergence Form - 2

Theorem ((Serrin, 1964), (Trudinger, 1967))

Let $u \in W^{1,p}_{loc}(\Omega) \cap L^\infty(\Omega)$*,* $p > 1$*, be a weak supersolution to*

$$\operatorname{div} \mathbf{A}(., u, \nabla u) + B(., u, \nabla u) = 0 \quad \text{in } \Omega, ,$$

with test functions from $W^{1,p}_{\mathrm{loc}}(\Omega) \cap L^\infty(\Omega)$*. Assume* $\|u\|_\infty \leq M$*. Then there are a positive constant* $c_1 = c_1(p, d, c_0, d_0, M)$ *and* $m : (0, \infty) \to \mathbb{R}$*,*
$0 < m(\rho) \leq c_2 \rho^{\frac{p}{p-1}}$ *such that for any ball* $B_{3\rho}(x_0) \subset \Omega$

$$\Big(\tfrac{1}{|B_{2\rho}(x_0)|} \int\limits_{B_{2\rho}(x_0)} |u|^{p-1} \Big)^{\frac{1}{p-1}} \leq c_1 \big\{ \inf_{B_\rho(x_0)} u + m(\rho) \big\} . \qquad (13)$$

Here $c_2 > 0$ *depends on the norms of* c_1, c_2, d_1, d_2 *plus on other data.*

Parabolic Equations in Divergence Form - 1

Theorem (Pini 1954, Hadamard 1955, Moser 1964/1967, Auchmuty-Bao 1994)

Let $u \in C^\infty((0,\infty) \times \mathbb{R}^d)$ be a non-negative solution of the heat equation, i.e. $\frac{\partial}{\partial t} u - \Delta u = 0$. Then

$$u(t_1, x) \leq u(t_2, y) \left(\tfrac{t_2}{t_1}\right)^{d/2} e^{\frac{|y-x|^2}{4(t_2-t_1)}}, \quad x, y \in \mathbb{R}^d, t_2 > t_1 \,., \tag{14}$$

$$\sup_{|x|\leq\rho, \theta_1^- < t < \theta_2^-} u(t,x) \leq c \inf_{|x|\leq\rho, \theta_1^+ < t < \theta_2^+} u(t,x) \tag{15}$$

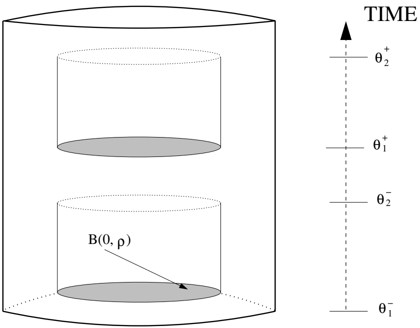
for non-negative solutions to the heat equation in $(0, \theta_2^+) \times B_R(0)$ as long as $\theta_2^- < \theta_1^+$. Here $c = c(d, \theta_1^-, \theta_2^-, \theta_1^+, \theta_2^+, \rho, R)$.

Estimate (15) can be illuminated as follows. Think of $u(t,x)$ as the amount of heat at time t in point x. Assume $u(t,x) \geq 1$ for some point $x \in B_\rho(0)$ at time $t \in (\theta_1^-, \theta_2^-)$. Then, after some waiting time, i.e. for $t > \theta_1^+$ $u(t,x)$ will be greater some constant c in all of the ball $B_\rho(0)$.

Parabolic Equations in Divergence Form - 1

$$\sup_{|x|\leq\rho,\theta_1^-<t<\theta_2^-} u(t,x) \leq c \inf_{|x|\leq\rho,\theta_1^+<t<\theta_2^+} u(t,x) \qquad (16)$$

for non-negative solutions to the heat equation in $(0,\theta_2^+)\times B_R(0)$ as long as $\theta_2^- < \theta_1^+$. Here $c = c(d,\theta_1^-,\theta_2^-,\theta_1^+,\theta_2^+,\rho,R)$.



If $\theta_2^- - \theta_1^- \approx \theta_2^+ - \theta_1^+ \approx c\rho^2$ and $\rho = R/2$ then c is universal.

Note:
It is necessary to wait some little amount of time for the phenomenon to occur since there is a sequence of solutions u_n satisfying $\frac{u_n(1,0)}{u_n(1,x)} \to 0$ for $n \to \infty$

Parabolic Equations in Divergence Form - 2

Assume $(t,x) \mapsto A(t,x) = \big(a_{ij}(t,x)\big)_{i,j=1,\ldots,d}$ satisfies $a_{ij} \in L^\infty((0,\infty)\times\mathbb{R}^d)$ $(i,j=1,\ldots,d)$ and

$$\lambda|\xi|^2 \le a_{ij}(t,x)\xi_i\xi_j \le \lambda^{-1}|\xi|^2 \quad \forall (t,x)\in(0,\infty)\times\mathbb{R}^d, \xi\in\mathbb{R}^d\,. \tag{17}$$

for some $\lambda > 0$.

Theorem (Moser '64, '67, '71)

Assume $u \in L^\infty\big(0,T;L^2(B_R(0))\big) \cap L^2\big(0,T;H^1(B_R(0))\big)$ is a non-negative weak solution to the equation

$$u_t - \operatorname{div}\big(A(.,.)\nabla u\big) = 0 \quad \text{in } (0,T)\times B_R(0)\,. \tag{18}$$

Then for any choice of constants $0<\theta_1^- < \theta_2^- < \theta_1^+ < \theta_2^+$, $0<\rho<R$ there exists a positive constant c depending only on these constants and on the space dimension d such that (15) holds.

Parabolic Equations in Divergence Form - 3

- The above theorem implies Hölder regularity for the fundamental solution Γ (transition density) in a way similar to the elliptic setting. Recoverage of Nash's results from 1958.
- In [Aronson, 1967] Theorem 8 is used in order to prove sharp bounds on the fundamental solution $\Gamma(t, x; s, y)$ to the operator $\partial_t - \operatorname{div}\big(A(.,.)\nabla\big)$:

$$c_1(t-s)^{-d/2} e^{\frac{-c_2|x-y|^2}{|t-s|}} \leq \Gamma(t, x; s, y) \leq c_3(t-s)^{-d/2} e^{\frac{-c_4|x-y|^2}{|t-s|}} \,. \qquad (19)$$

  The constants $c_i > 0, i = 1, \ldots, 4$, depend only on d and λ.
- In [Fabes and Stroock, 1986] the technique of [Nash, 1958] is applied in order to prove (19). Moreover, they finally show that the results of [Nash, 1958] already imply Theorem 8.
- See also [Ferretti/Safonov 2001], [Safonov 2002] for an alternative approach.

Elliptic Equations in Nondivergence Form

Early contributions: [Nirenberg, 1953], [Cordes, 1956], [Landis, 1972].
Assume $(t,x) \mapsto A(t,x) = \big(a_{ij}(t,x)\big)_{i,j=1,\ldots,d}$ satisfies (17). Set $Q_{\theta,R}(t_0,x_0) = (t_0 + \theta R^2) \times B_R(x_0)$ and $Q_{\theta,R} = Q_{\theta,R}(0,0)$.

Theorem (Krylov/Safonov, 1980)

Let $\theta > 1$ and $R \leq 2$, $u \in W_2^{1,2}(Q_{\theta,R})$, $u \geq 0$ be such that

$$u_t - a_{ij} D_i D_j u = 0 \qquad \text{a.e. in } Q_{\theta,R}\,. \tag{20}$$

Then there is a constant C depending only on λ, θ, d such that

$$u(R^2, 0) \leq C u(\theta R^2, x) \quad \forall\, x \in B_{R/2}\,. \tag{21}$$

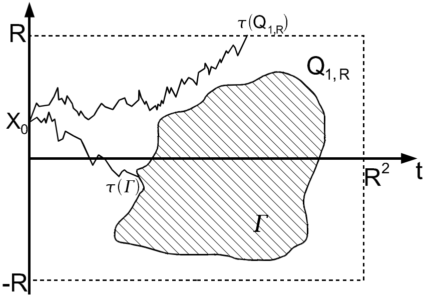
The constant C stays bounded as long as $(1-\theta)^{-1}$ and λ^{-1} stay bounded.

Elliptic Equations in Nondivergence Form - 2

One considers the diffusion process (X_t) associated to the operator $a_{ij}D_iD_j$ via the martingale problem. This process solves the following system of ordinary stochastic differential equations $dX_t = \sigma_t dB_t$. Here (B_t) is a $d-$dimensional Brownian motion and $\sigma_t^T \sigma_t = A$.

Assume that $\mathbb{P}(X_0 \leq \alpha R) = 1$ where $\alpha \in (0,1)$. Let $\Gamma \subset Q_{1,R}$ be a closed set satisfying $|\Gamma| \geq \varepsilon |Q_{1,R}|$ for some $\varepsilon > 0$. For a set $M \subset (0,\infty) \times \mathbb{R}^d$ let us denote the time when (X_t) hits the boundary ∂M by $\tau(M) = \inf\{t > 0; (t, X_t) \in \partial M\}$. The key idea in the proof is to show that there is $\delta > 0$ depending only on $d, \lambda, \alpha, \varepsilon$ such that

$$\mathbb{P}\big(\tau(\Gamma) < \tau(Q_{1,R})\big) \geq \delta \quad \forall\, R \in (0,1)\,.$$



Degenerate Operators of Second Order - 1

A possible characterization can be made with the help of the following properties:

divergence form	or	**non-divergence form**
elliptic	or	**parabolic**
x-**degeneration of coeff.**	or	u- **or** ∇u-**degeneration of coeff.**

Two examples (elliptic, divergence form, ∇u-degeneration of coefficients):

1. p-Laplace equation $\operatorname{div}\left(|\nabla u|^{p-2}\nabla u\right) = 0$ covered by [Serrin, 1964].
2. Minimal surface equation $\operatorname{div}\left(\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}\right) = 0$ not covered by [Serrin, 1964], [Trudinger, 1967] but in [Trudinger, 1981].

Degenerate Operators of Second Order - 2

The analogous situation for degenerate parabolic equations is very different, even for the model equation $u_t - \operatorname{div}\big(|\nabla u|^{p-2}\nabla u\big) = 0, p > 2$.

- Classical parabolic Harnack inequality fails.
- Hölder regularity is possible, here $\theta_2^- - \theta_1^- \approx \theta_2^+ - \theta_1^+ \approx cu^{2-p}$ (!).
- A u dependend, time-intrinsic geometry is needed (see [DiBenedetto 1993]).

New development:
In [DiBenedetto/Gianazza/Vespri 2006], for quasilinear versions of

$$u_t - \partial_i\big(|\nabla u|^{p-2} a_{i,j}(t,x)\partial_j u\big) = 0, \quad p > 2,$$

the following Harnack inequality is established:

$$u(t_0, x_0) \leq c_1 \inf_{x\in B_\rho(x_0)} u\Big(t_0 + \big(\tfrac{c_2}{u(t_0,x_0)}\big)^{p-2}\rho^p, x\Big) .$$

Degenerate Operators of Second Order - 3

[Fabes/Kenig/Serapioni, 1982]: Harnack inequality assuming

$$\Lambda(x)/\lambda(x) \leq C, \qquad \text{and } \lambda \in A_2, \text{ i.e. for all balls } B \subset \mathbb{R}^d$$

$$\Big(\frac{1}{|B|}\int_B \lambda(x)\,dx\Big)\Big(\frac{1}{|B|}\int_B (\lambda(x))^{-1}\,dx\Big) \leq C\,. \tag{22}$$

Ingredients: Weighted Poincaré inequalities plus Moser's iteration technique.

If $\Lambda(x)/\lambda(x)$ may be unbounded one cannot say in general whether a Harnack inequality or local Hölder a-priori estimates hold.

- positive: [Trudinger, 1971],
  negative: [Franchi, Serapioni, Serra Cassano, 1998].
- parabolic equations (Harnack-type inequalities): [Kruzhkov/Kolodii, 1977], [Chiarenza/Serapioni, 1984, 1987]

In case (22) plus $\lambda(2B) \leq c\lambda(B)$, $\Lambda(2B) \leq c\Lambda(B)$ certain Poincaré and Sobolev inequalities hold, see [Chanillo/Wheeden, 1986, 1988] Harnack-type inequalities.

Important Issues omited:

- Harnack inequality of [Li/Yau, 1986].
- Perelman's use of Harnack inequalities. See [R. Müller, 2006].
- Viscosity solutions, fully non-linear pde's, [Caffarelli, 1989]
- Let (M, g) be a smooth, geodesically complete Riemannian manifold of dimension d. Let $r_0 > 0$. Then the two properties

$$\big|B(x,2r)\big| \leq c_1 \big|B(x,r)\big|, \quad 0 < r < r_0, x \in M\,, \tag{23}$$

$$\int\limits_{B(x,r)} |f - f_{x,r}|^2 \leq c_2 r^2 \int\limits_{B(x,2r)} |\nabla f|^2, \quad 0 < r < r_0, x \in M, f \in C^\infty(M)\,, \tag{24}$$

  together are equivalent to the parabolic Harnack inequality. [Grigor'yan, 1991], [Saloff-Coste, 1992].
- Inhomogeneous non-local operators.