Scaling random walks on critical random trees and graphs

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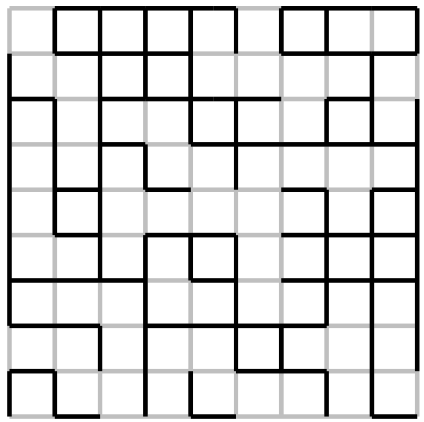
1. MOTIVATING EXAMPLES

RANDOM WALK ON PERCOLATION CLUSTERS

Bond percolation on integer lattice $\mathbb{Z}^d$ ($d \geq 2$), parameter $p > p_c$.
e.g. $p = 0.54$,



Given a configuration ω, let X^ω be the (continuous time) simple random walk on the unique infinite cluster – the 'ant in the labyrinth' [de Gennes 1976]. For $\mathbb{P}_p$-a.e. realisation of the environment,

$$q_t^\omega(x,y) = \frac{P_x^\omega(X_t^\omega = y)}{\mathrm{deg}_\omega(y)} \asymp c_1 t^{-d/2} e^{-c_2|x-y|^2/t}$$

for $t \geq |x-y| \vee S_x(\omega)$ [Barlow 2004].

RANDOM WALK ON PERCOLATION CLUSTERS

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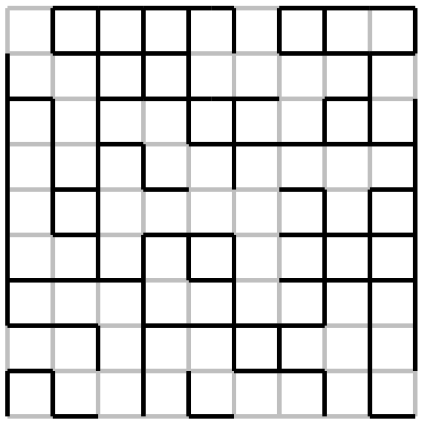


[Sidoravicius/Sznitman 2004, Biskup/Berger 2007, Mathieu/Piatnitski 2007] For $\mathbb{P}_p$-a.e. realisation of the environment

$$\left(n^{-1} X^{\omega}_{n^2 t}\right)_{t \geq 0} \to (B_{\sigma t})_{t \geq 0}$$

in distribution, where $\sigma \in (0, \infty)$ is a deterministic constant.

ANOMALOUS BEHAVIOUR AT CRITICALITY

At criticality, $p = p_c$, physicists conjectured that the associated random walks had an anomalous **spectral dimension** [Alexander/Orbach 1982]: for every $d \geq 2$,

$$d_s = -2 \lim_{n\to\infty} \frac{\log P_x^\omega(X_{2n}^\omega = x)}{\log n} = \frac{4}{3}.$$

[Kesten 1986] constructed the law of the **incipient infinite cluster** in two dimensions, i.e.

$$\mathbb{P}_{\mathrm{IIC}} = \lim_{n\to\infty} \mathbb{P}_{p_c}\left(\cdot \middle| 0 \leftrightarrow \partial[-n,n]^2 \right),$$

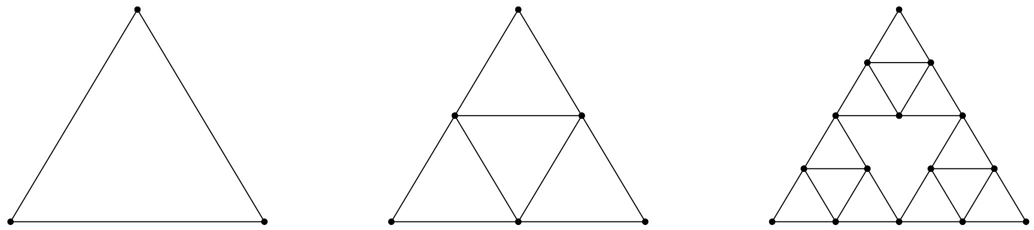
and showed that random walk on the IIC in two dimensions satisfies:

$$\left(n^{-\frac{1}{2}+\varepsilon} X_n^{\mathrm{IIC}} \right)_{n\geq 0}$$

is tight – this shows the walk is **subdiffusive**.

ANOMALOUS DIFFUSIONS ON FRACTALS

Interest from physicists [Rammal/Toulouse 1983], and construction of diffusion on fractals such as the Sierpinski gasket:



[Barlow/Perkins 1988] constructed diffusion (see also [Kigami 1989]), and established **sub-Gaussian** heat kernel bounds:

$$q_t(x,y) \asymp c_1 t^{-d_s/2} \exp\left\{-c_2(|x-y|^{d_w}/t)^{\frac{1}{d_w-1}}\right\}.$$

NB. $d_s/2 = d_f/d_w$ – the **Einstein relation**. More robust techniques applicable to random graphs since developed.

THE '$d=\infty$' CASE

Let T be a d-regular tree. Then $p_c = 1/d$. We can define

$$\mathbb{P}_{\mathrm{IIC}} = \lim_{n\to\infty} \mathbb{P}_{p_c}(\,\cdot\,|\,\rho \leftrightarrow \text{generation } n)\,,$$

e.g. [Kesten 1986].

[Barlow/Kumagai 2006] show AO conjecture holds for $\mathbb{P}_{\mathrm{IIC}}$-a.e. environment, $\mathbb{P}_{\mathrm{IIC}}$-a.s. subdiffusivity

$$\lim_{n\to\infty} \frac{\log E_\rho^{\mathrm{IIC}}(\tau_n)}{\log n} = 3,$$

and sub-Gaussian annealed heat kernel bounds.

Similar techniques used/results established for oriented percolation in high dimensions [Barlow/Jarai/Kumagai/Slade 2008], invasion percolation on a regular tree [Angel/Goodman/den Hollander/Slade 2008], see also [Kumagai/Misumi 2008].

PROGRESS IN HIGH DIMENSIONS

Law $\mathbb{P}_{\mathrm{IIC}}$ of the **incipient infinite cluster** in high dimensions constructed in [van der Hofstad/Járai 2004].

Fractal dimension (in intrinsic metric) is 2. Unique backbone, scaling limit is Brownian motion. Scaling limit of IIC is related to **integrated super-Brownian excursion** [Kozma/Nachmias 2009, Heydenreich/van der Hofstad/Hulshof/Miermont 2013, Hara/Slade 2000].

Random walk on IIC satisfies AO conjecture ($d_s = 4/3$), and behaves subdiffusively [Kozma/Nachmias 2009], e.g. $\mathbb{P}_{\mathrm{IIC}}$-a.s.,

$$\lim_{n\to\infty} \frac{\log E_0^\omega(\tau_n)}{\log n} = 3.$$

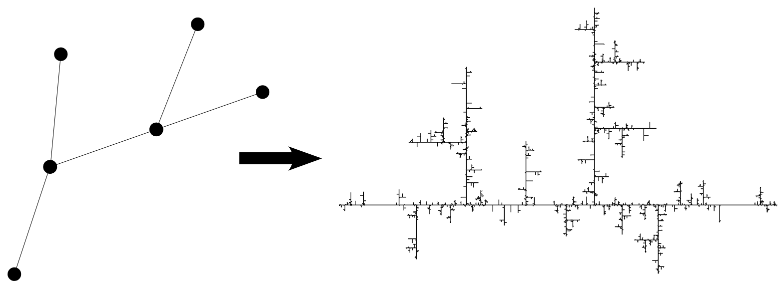
See also [Heydenreich/van der Hofstad/Hulshof 2014].

CRITICAL GALTON-WATSON TREES

Let T_n be a Galton-Watson tree with a critical (mean 1), aperiodic, finite variance offspring distribution, conditioned to have n vertices, then

$$n^{-1/2} T_n \to \mathcal{T},$$

where $\mathcal{T}$ is (up to a deterministic constant) the **Brownian continuum random tree (CRT)** [Aldous 1993], also [Duquesne/Le Gall 2002].



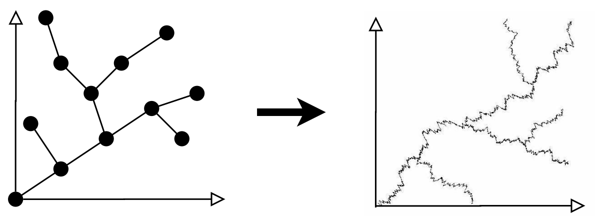
Result includes various combinatorial random trees. Similar results for infinite variance case.

CRITICAL BRANCHING RANDOM WALK

Given a graph tree T with root ρ, let $(\delta(e))_{e\in E(T)}$ be a collection of edge-indexed, i.i.d. random variables. We can use this to embed the vertices of T into $\mathbb{R}^d$ by:

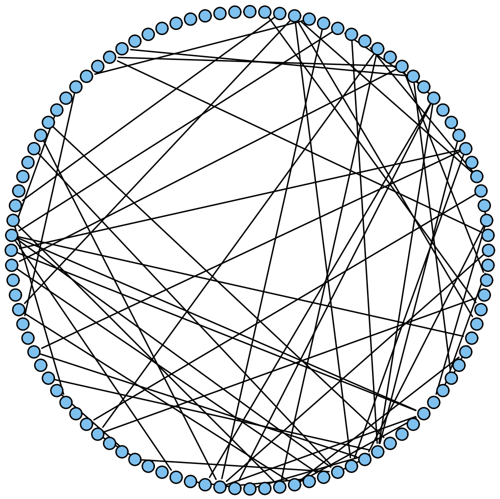
$$v \mapsto \sum_{e\in[[\rho,v]]} \delta(e).$$

If T_n are critical Galton-Watson trees with finite exponential moment offspring distribution, and $\delta(e)$ are centred and satisfy $\mathbb{P}(\delta(e) > x) = o(x^{-4})$, then the corresponding **branching random walk** has an integrated super-Brownian excursion scaling limit [Janson/Marckert 2005].



CRITICAL ERDŐS-RÉNYI RANDOM GRAPH

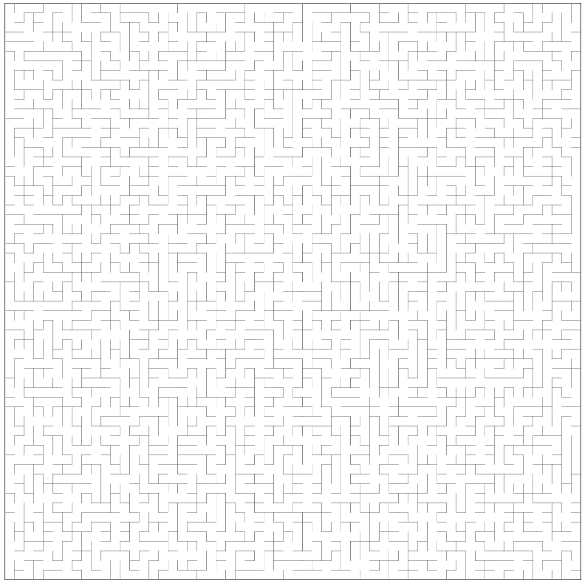
$G(n,p)$ is obtained via bond percolation with parameter p on the complete graph with n vertices. We concentrate on critical window: $p = n^{-1} + \lambda n^{-4/3}$. e.g. $n = 100$, $p = 0.01$:



All components have:
- size $\Theta(n^{2/3})$ and surplus $\Theta(1)$ [Erdős/Rényi 1960], [Aldous 1997],
- diameter $\Theta(n^{1/3})$ [Nachmias/Peres 2008].

Moreover, asymptotic structure of components is related to the Brownian CRT [Addario-Berry/Broutin/Goldschmidt 2010].

TWO-DIMENSIONAL UNIFORM SPANNING TREE



Let $\Lambda_n := [-n,n]^2 \cap \mathbb{Z}^2$.

A subgraph of the lattice is a **spanning tree** of Λ_n if it connects all vertices and has no cycles.

Let $\mathcal{U}^{(n)}$ be a spanning tree of Λ_n selected uniformly at random from all possibilities.

The UST on $\mathbb{Z}^2$, $\mathcal{U}$, is then the local limit of $\mathcal{U}^{(n)}$.

Almost-surely, $\mathcal{U}$ is a spanning tree of $\mathbb{Z}^2$. (Forest for $d > 4$.) Fractal dimension 8/5. SLE-related scaling limit.

[Aldous, Barlow, Benjamini, Broder, Häggström, Kirchoff, Lawler, Lyons, Masson, Pemantle, Peres, Schramm, Werner, Wilson,. . .]

RANDOM WALKS ON RANDOM TREES AND GRAPHS AT CRITICALITY

In the following, the aim is to:

• Introduce techniques for showing random walks on (some of) the above random graphs converge to a diffusion on a fractal;

• Study the properties of these scaling limits.

Brief outline:

2. Gromov-Hausdorff and related topologies
3. Dirichlet forms and diffusions on real trees
4. Traces and time change
5. Scaling random walks on graph trees

...

6. Fusing and the critical random graph
7. Spatial embeddings
8. Local times and cover times

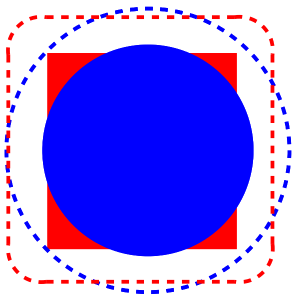
2. GROMOV-HAUSDORFF AND RELATED TOPOLOGIES

HAUSDORFF DISTANCE

The **Hausdorff distance** between two non-empty compact subsets K and K' of a metric space (M, d_M) is defined by

$$d_M^H(K, K') := \max\left\{\sup_{x \in K} d_M(x, K'), \sup_{x' \in K'} d_M(x', K)\right\}$$
$$= \inf\left\{\varepsilon > 0 : \; K \subseteq K'_\varepsilon, \; K' \subseteq K_\varepsilon\right\},$$

where $K_\varepsilon := \{x \in M : \; d_M(x, K) \leq \varepsilon\}$.



If (M, d_M) is complete (resp. compact), then so is the collection of non-empty compact subsets equipped with this metric.

GROMOV-HAUSDORFF DISTANCE

For two non-empty compact metric spaces (K, d_K), $(K', d_{K'})$, the Gromov-Hausdorff distance between them is defined by setting

$$d_{GH}(K, K') := \inf d_M^H(\phi(K), \phi'(K')),$$

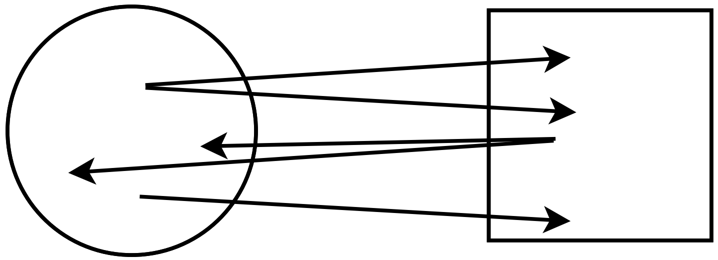
where the infimum is taken over all metric space (M, d_M) and isometric embeddings $\phi : K \to M$, $\phi' : K' \to M$.

The function d_{GH} is a metric on the collection of (isometry classes of) non-empty compact metric spaces. Moreover, the resulting metric space is complete and separable.

For background, see [Gromov 2006, Burago/Burago/Ivanov 2001].

CORRESPONDENCES

A correspondence $\mathcal{C}$ is a subset of $K \times K'$ such that for every $x \in K$ there exists an $x' \in K'$ such that $(x, x') \in \mathcal{C}$, and vice versa.



The distortion of a correspondence is:

$$\operatorname{dis} \mathcal{C} = \sup \left\{ |d_K(x,y) - d_{K'}(x',y')| : (x,x'), (y,y') \in \mathcal{C} \right\}.$$

An alternative characterisation of the Gromov-Hausdorff distance is then:

$$d_{GH}(K, K') = \frac{1}{2} \inf \operatorname{dis} \mathcal{C}.$$

EXAMPLE: CONVERGENCE OF GW TREES

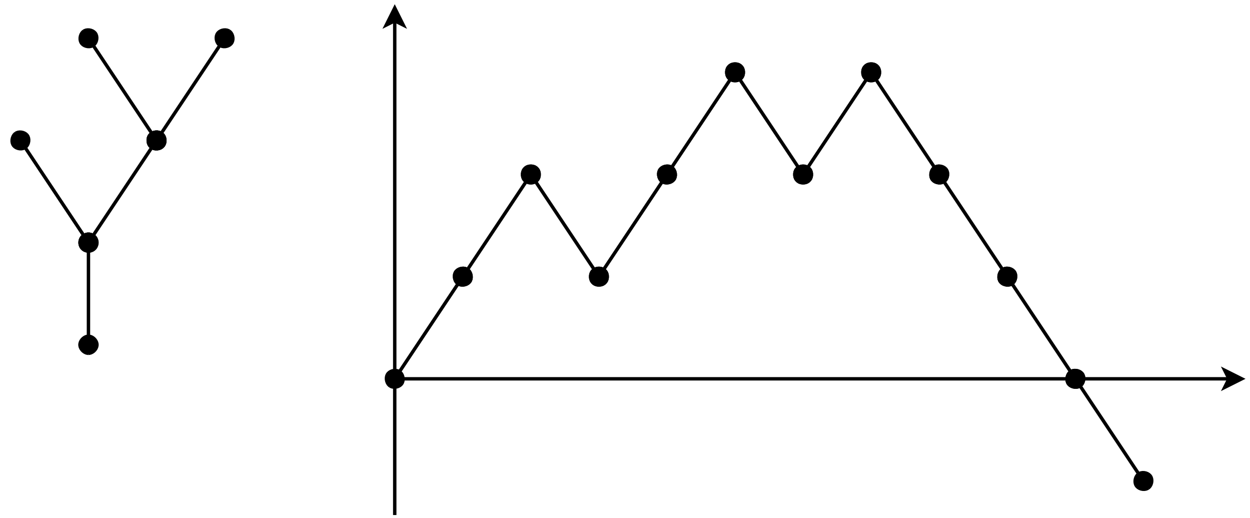
Let T_n be a Galton-Watson tree with a critical (mean 1), aperiodic, finite variance σ^2 offspring distribution, conditioned to have n vertices, then

$$\left(T_n, \frac{\sigma}{2n^{1/2}} d_{T_n}\right) \to (\mathcal{T}, d_{\mathcal{T}})$$

in distribution, with respect to the Gromov-Hausdorff topology. The limiting tree is the Brownian continuum random tree, cf. [Aldous 1993].

DISCRETE CONTOUR FUNCTION

Given an ordered graph tree T, its contour function measures the height of a particle that traces the 'contour' of the tree at unit speed from left to right.



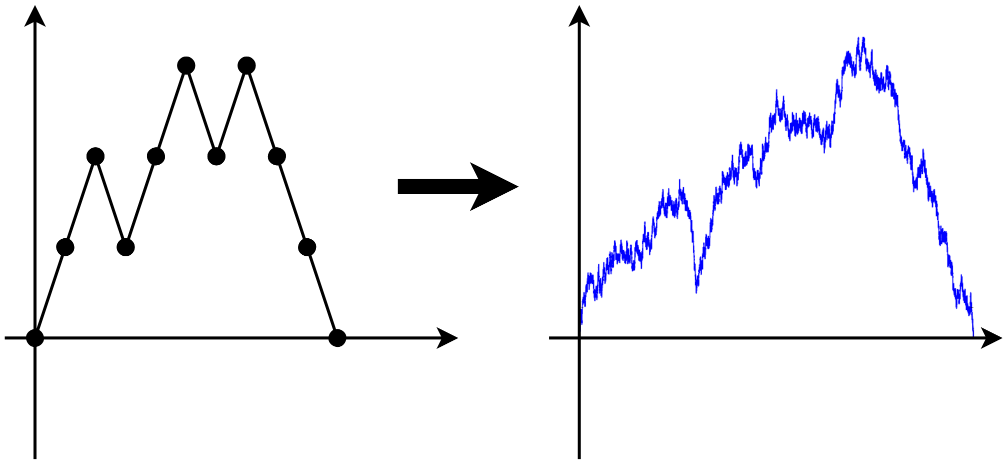
e.g. If a GW tree has a geometric, parameter $\frac{1}{2}$, distribution, then the contour function is precisely a random walk stopped at the first time it hits -1 [Harris 1952]. Conditioning tree to have n vertices equivalent to conditioning the walk to hit -1 at time $2n-1$.

CONVERGENCE OF CONTOUR FUNCTIONS

Let $(C_n(t))_{t\in[0,2n-1]}$ be the contour function of T_n. Then

$$\left(\frac{\sigma}{2n^{1/2}}C_{2(n-1)t}\right)_{t\in[0,1]} \to (B_t)_{t\in[0,1]},$$

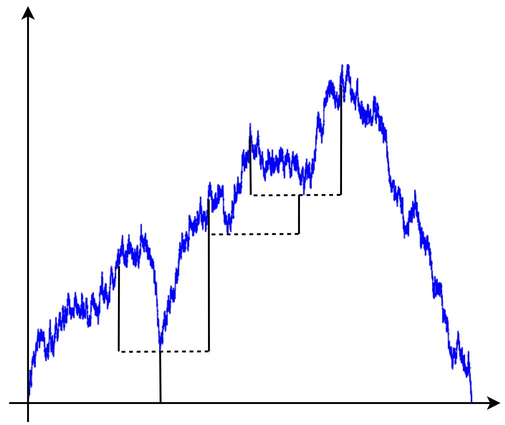
in distribution in the space $C([0,1],\mathbb{R})$, where the limit process is Brownian excursion normalised to have length one.



See [Marckert/Mokkadem 2003] for a nice general proof.

EXCURSIONS AND REAL TREES

Consider an excursion $(e(t))_{t\in[0,1]}$ – that is, a continuous function that satisfies $e(0) = e(1) = 0$ and is strictly positive for $t \in (0,1)$.



Define a distance on [0,1] by setting

$$d_e(s,t) := e(s) + e(t) - 2 \min_{r\in[s\wedge t, s\vee t]} e(r).$$

Then we obtain a (compact) real tree (see definition below) by setting $\mathcal{T}_e = [0,1]/\sim$, where $s \sim t$ iff $d_e(s,t) = 0$. [Duquesne/Le Gall 2004]

CONVERGENCE IN GH TOPOLOGY

Let $\mathcal{T} = \mathcal{T}_B$ – this is the Brownian continuum random tree.

Since $C([0,1],\mathbb{R})$ is separable, we can couple (rescaled) contour processes so that they converge almost-surely. Consider correspondence between T_n and $\mathcal{T}$ given by

$$\mathcal{C} = \{([\lceil 2(n-1)t \rceil]_n, [t]) : \, t \in [0,1]\},$$

where $[t]$ is the equivalence class of t with respect to $\sim$, and similarly for $[t]_n$. This satisfies

$$\operatorname{dis} \mathcal{C} \leq 4 \left\| \frac{\sigma}{2n^{1/2}} C_{2(n-1)\cdot} - B \right\|_\infty \to 0.$$

Hence

$$d_{GH}\left(\left(T_n, \frac{\sigma}{2n^{1/2}} d_{T_n}\right), (\mathcal{T}, d_{\mathcal{T}})\right) \leq 2 \left\| \frac{\sigma}{2n^{1/2}} C_{2(n-1)\cdot} - B \right\|_\infty \to 0.$$

INCORPORATING POINTS AND MEASURE

For two non-empty compact pointed metric probability measure spaces (K, d_K, μ_K, ρ_K), $(K', d_{K'}, \mu_{K'}, \rho_{K'})$, we define a distance by setting $d_{GHP}(K, K')$ to be equal to

$$\inf \left\{ d_M(\phi(\rho_K), \phi'(\rho_{K'})) + d_M^H(\phi(K), \phi'(K')) + d_M^P(\mu_K \circ \phi^{-1}, \mu_{K'} \circ \phi'^{-1}) \right\},$$

where the infimum is taken over all metric space (M, d_M) and isometric embeddings $\phi : K \to M$, $\phi' : K' \to M$. Here d_M^P is the Prohorov metric between probability measures on M, i.e.

$$d_M^P(\mu, \nu) = \inf\{\varepsilon : \mu(A) \leq \nu(A_\varepsilon) + \varepsilon,\ \nu(A) \leq \mu(A_\varepsilon) + \varepsilon,\ \forall A\}.$$

The function d_{GHP} is a metric on the collection of (measure and root preserving isometry classes of) non-empty compact pointed metric probability measure spaces. (Again, complete and separable.) [Abraham/Delmas/Hoscheit 2013]

EXAMPLE: GHP CONVERGENCE OF GW TREES

Let T_n be a Galton-Watson tree with a critical (mean 1), aperiodic, finite variance σ^2 offspring distribution, conditioned to have n vertices. Let μ_{T_n} be the uniform probability measure on T_n, and ρ_{T_n} its root. Then

$$\left(T_n, \frac{\sigma}{2n^{1/2}} d_{T_n}, \frac{1}{n}\mu_{T_n}, \rho_{T_n}\right) \to (\mathcal{T}, d_{\mathcal{T}}, \mu_{\mathcal{T}}, \rho_{\mathcal{T}})$$

in distribution, with respect to the topology induced by d_{GHP}. The limiting tree is the Brownian continuum random tree. In the excursion construction $\rho_{\mathcal{T}} = [0]$, and

$$\mu_{\mathcal{T}} = \lambda \circ p^{-1},$$

where λ is Lebesgue measure on $[0,1]$ and $p : t \mapsto [t]$ is the canonical projection.

PROOF IDEA

Consider two length one excursions e and f. As before, define a correspondence $\mathcal{C} = \{([t]_e, [t]_f) : t \in [0,1]\}$, and note that $\operatorname{dis}\mathcal{C} \leq 4\|e-f\|_\infty$. Let $M = \mathcal{T}_e \sqcup \mathcal{T}_f$, with metric d_M equal to $d_{\mathcal{T}_e}$, $d_{\mathcal{T}_f}$ on $\mathcal{T}_e$, $\mathcal{T}_f$ resp., and

$$d_M(x,x') = \inf\{d_{\mathcal{T}_e}(x,y) + \tfrac{1}{2}\operatorname{dis}\mathcal{C} + d_{\mathcal{T}_f}(y',x') : (y,y') \in \mathcal{C}\},$$

for $x \in \mathcal{T}_e$, $x' \in \mathcal{T}_f$. Then

$$d_M([0]_e, [0]_f) = \tfrac{1}{2}\operatorname{dis}\mathcal{C} = d_M^H(\mathcal{T}_e, \mathcal{T}_f).$$

Moreover, if A is a measurable subset of $\mathcal{T}_e$ and $B = p_f(p_e^{-1}(A)) \subseteq \mathcal{T}_f$, then $B \subseteq A_\varepsilon$ for $\varepsilon > \frac{1}{2}\operatorname{dis}\mathcal{C}$ and

$$\mu_{\mathcal{T}_e}(A) \leq \mu_{\mathcal{T}_f}(B) \leq \mu_{\mathcal{T}_e}(A_\varepsilon).$$

By symmetry, it follows that

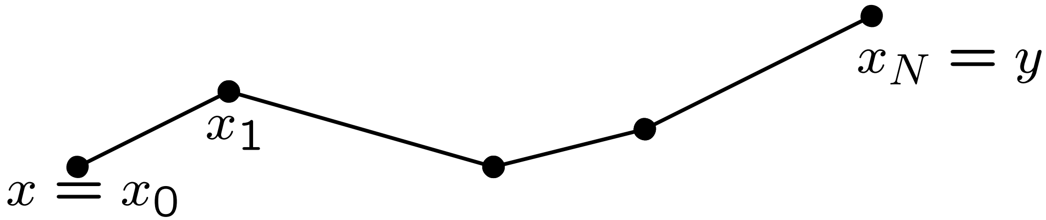
$$d_M^P(\mu_{\mathcal{T}_e}, \mu_{\mathcal{T}_f}) \leq \tfrac{1}{2}\operatorname{dis}\mathcal{C}.$$

3. DIRICHLET FORMS AND DIFFUSIONS ON REAL TREES

REAL TREES

A **compact real tree** $(\mathcal{T}, d_{\mathcal{T}})$ is an arcwise-connected compact topological space containing no subset homeomorphic to the circle. Moreover, the unique arc between two points x, y is isometric to $[0, d_{\mathcal{T}}(x, y)]$. (cf. compact metric trees [Athreya/Lohr/Winter].)

In particular, the metric $d_{\mathcal{T}}$ on a real tree is additive along paths, i.e. if $x = x_0, x_1, \ldots, x_N = y$ appear in order along an arc



then

$$d_{\mathcal{T}}(x, y) = \sum_{i=1}^{N} d_{\mathcal{T}}(x_{i-1}, x_i).$$

APPROACH FOR CONSTRUCTING A DIFFUSION

Given a compact real tree $(\mathcal{T}, d_{\mathcal{T}})$ and finite Borel measure $\mu^{\mathcal{T}}$ of full support, we aim to construct a quadratic form $\mathcal{E}^{\mathcal{T}}$ that is a local, regular Dirichlet form on $L^2(\mu^{\mathcal{T}})$.

Then, through the standard association

$$\mathcal{E}^{\mathcal{T}}(f,g) = -\int_{\mathcal{T}} (\Delta_{\mathcal{T}} f) g d\mu^{\mathcal{T}} \Leftrightarrow P_t^{\mathcal{T}} = e^{t\Delta_{\mathcal{T}}},$$

define Brownian motion on $(\mathcal{T}, d_{\mathcal{T}}, \mu^{\mathcal{T}})$ to be the Markov process with generator $\Delta_{\mathcal{T}}$.

We follow the construction of [Athreya/Eckhoff/Winter 2013], see also [Krebs 1995] and [Kigami 1995].

DIRICHLET FORM DEFINITION

Let $(\mathcal{T}, d_{\mathcal{T}})$ be a compact real tree, and $\mu^{\mathcal{T}}$ be a finite Borel measure of full support. A **Dirichlet form** $(\mathcal{E}^{\mathcal{T}}, \mathcal{F}^{\mathcal{T}})$ on $L^2(\mu^{\mathcal{T}})$ is a bilinear map $\mathcal{F}^{\mathcal{T}} \times \mathcal{F}^{\mathcal{T}} \to \mathbb{R}$ that is:

- symmetric, i.e. $\mathcal{E}^{\mathcal{T}}(f,g) = \mathcal{E}^{\mathcal{T}}(g,f)$,
- non-negative, i.e. $\mathcal{E}^{\mathcal{T}}(f,f) \geq 0$,
- Markov, i.e. if $f \in \mathcal{F}^{\mathcal{T}}$, then so is $\bar{f} := (0 \vee f) \wedge 1$ and $\mathcal{E}^{\mathcal{T}}(\bar{f},\bar{f}) \leq \mathcal{E}^{\mathcal{T}}(f,f)$,
- closed, i.e. $\mathcal{F}^{\mathcal{T}}$ is complete w.r.t.
$$\mathcal{E}_1^{\mathcal{T}}(f,f) := \mathcal{E}^{\mathcal{T}}(f,f) + \int_{\mathcal{T}} f(x)^2 \mu^{\mathcal{T}}(dx),$$
- dense, i.e. $\mathcal{F}^{\mathcal{T}}$ is dense in $L^2(\mu^{\mathcal{T}})$.

It is **regular** if $\mathcal{F}^{\mathcal{T}} \cap C(\mathcal{T})$ is dense in $\mathcal{F}^{\mathcal{T}}$ w.r.t. $\mathcal{E}_1^{\mathcal{T}}$, and dense in $C(\mathcal{T})$ w.r.t. $\|\cdot\|_\infty$.

ASSOCIATION WITH SEMIGROUP

[Fukushima/Oshima/Takeda 2011, Sections 1.3-1.4] Let $(P_t^{\mathcal{T}})_{t\geq 0}$ be a strongly continuous $\mu^{\mathcal{T}}$-symmetric Markovian semigroup on $L^2(\mu^{\mathcal{T}})$. For $f \in L^2(\mu^{\mathcal{T}})$, define

$$\mathcal{E}_t^{\mathcal{T}}(f,f) := t^{-1}\int_{\mathcal{T}}(f - P_t^{\mathcal{T}} f) f d\mu^{\mathcal{T}}.$$

This is non-negative and non-decreasing in t. Let

$$\mathcal{E}^{\mathcal{T}}(f,f) := \lim_{t\downarrow 0} \mathcal{E}_t^{\mathcal{T}}(f,f), \qquad \mathcal{F}^{\mathcal{T}} := \left\{ f \in L^2(\mu^{\mathcal{T}}) : \lim_{t\downarrow 0} \mathcal{E}_t^{\mathcal{T}}(f,f) < \infty \right\}.$$

Then $(\mathcal{E}^{\mathcal{T}}, \mathcal{F}^{\mathcal{T}})$ is a Dirichlet form on $L^2(\mu^{\mathcal{T}})$. Moreover, if $\Delta_{\mathcal{T}}$ is the infinitesimal generator of $(P_t^{\mathcal{T}})_{t\geq 0}$, then $\mathcal{D}(\Delta_{\mathcal{T}}) \subseteq \mathcal{F}^{\mathcal{T}}$, $\mathcal{D}(\Delta_{\mathcal{T}})$ is dense in $L^2(\mu^{\mathcal{T}})$ and

$$\mathcal{E}^{\mathcal{T}}(f,g) = -\int_{\mathcal{T}} (\Delta_{\mathcal{T}} f) g d\mu^{\mathcal{T}}, \qquad \forall f \in \mathcal{D}(\Delta_{\mathcal{T}}),\ g \in \mathcal{F}^{\mathcal{T}}.$$

Conversely, if $(\mathcal{E}^{\mathcal{T}}, \mathcal{F}^{\mathcal{T}})$ is a Dirichlet form on $L^2(\mu^{\mathcal{T}})$, then there exists a strongly continuous $\mu^{\mathcal{T}}$-symmetric Markovian semigroup on $L^2(\mu^{\mathcal{T}})$ whose generator satisfies the above.

DIRICHLET FORMS ON GRAPHS

Let $G = (V(G), E(G))$ be a finite graph. Let $\lambda^G = (\lambda^G_e)_{e \in E(G)}$ be a collection of edge weights, $\lambda^G_e \in (0, \infty)$.

Define a quadratic form on G by setting

$$\mathcal{E}^G(f,g) = \frac{1}{2} \sum_{x,y: x \sim y} \lambda^G_{xy} \left(f(x) - f(y)\right) \left(g(x) - g(y)\right).$$

Note that, for any finite measure μ^G on $V(G)$ (of full support), $\mathcal{E}^G$ is a Dirichlet form on $L^2(\mu^G)$, and

$$\mathcal{E}^G(f,g) = - \sum_{x \in V(G)} (\Delta_G f)(x) g(x) \mu^G(\{x\}),$$

where

$$(\Delta_G f)(x) := \frac{1}{\mu^G(\{x\})} \sum_{y:\, y \sim x} \lambda^G_{xy} (f(y) - f(x)).$$

A FIRST EXAMPLE FOR A REAL TREE

For $(\mathcal{T}, d_{\mathcal{T}}) = ([0,1], \text{Euclidean})$ and μ be a finite Borel measure of full support on $[0,1]$. Let λ be Lebesgue measure on $[0,1]$, and define

$$\mathcal{E}(f,g) = \int_0^1 f'(x)g'(x)\lambda(dx), \qquad \forall f,g \in \mathcal{F},$$

where $\mathcal{F} = \{f \in C([0,1]) : f$ is abs. cont. and $f' \in L^2(\lambda)\}$. Then $(\mathcal{E}, \mathcal{F})$ is a regular Dirichlet form on $L^2(\mu)$. Note that

$$\mathcal{E}(f,g) = -\int_0^1 (\Delta f)(x)g(x)\mu(dx), \qquad \forall f \in \mathcal{D}(\Delta),\ g \in \mathcal{F},$$

where $\Delta f = \frac{d}{d\mu}\frac{df}{dx}$, and $\mathcal{D}(\Delta)$ contains those f such that: f' exists and df' is abs. cont. w.r.t. μ, $\Delta f \in L^2(\mu)$, and $f'(0) = f'(1) = 0$.

If $\mu = \lambda$, then the Markov process naturally associated with Δ is reflected Brownian motion on $[0,1]$.

GRADIENT ON REAL TREES

Let $(\mathcal{T}, d_{\mathcal{T}})$ be a compact real tree, with root $\rho_{\mathcal{T}}$.

Let $\lambda^{\mathcal{T}}$ be the 'length measure' on $\mathcal{T}$, and define orientation-sensitive integration with respect to $\lambda^{\mathcal{T}}$ by

$$\int_x^y g(z)\lambda^{\mathcal{T}}(dz) = \int_{b_{\mathcal{T}}(\rho_{\mathcal{T}},x,y)}^y g(z)\lambda^{\mathcal{T}}(dz) - \int_{b_{\mathcal{T}}(\rho_{\mathcal{T}},x,y)}^x g(z)\lambda^{\mathcal{T}}(dz).$$

Write

$$\mathcal{A} = \{f \in C(\mathcal{T}) : f \text{ is locally absolutely continuous}\}.$$

Proposition. If $f \in \mathcal{A}$, then there exists a unique function $g \in L^1_{\mathrm{loc}}(\lambda^{\mathcal{T}})$ such that

$$f(y) - f(x) = \int_x^y g(z)\lambda^{\mathcal{T}}(dz).$$

We say $\nabla_{\mathcal{T}} f = g$.

DIRICHLET FORMS ON REAL TREES

Let $(\mathcal{T}, d_{\mathcal{T}}, \rho_{\mathcal{T}})$ be a compact, rooted real tree, and $\mu^{\mathcal{T}}$ a finite Borel measure on $\mathcal{T}$ with full support. Define

$$\mathcal{F}^{\mathcal{T}} := \left\{ f \in \mathcal{A} : \nabla_{\mathcal{T}} f \in L^2(\lambda^{\mathcal{T}}) \right\} \left(\subseteq L^2(\mu^{\mathcal{T}}) \right).$$

For $f, g \in \mathcal{F}^{\mathcal{T}}$, set

$$\mathcal{E}^{\mathcal{T}}(f,g) = \int_{\mathcal{T}} \nabla_{\mathcal{T}} f(x) \nabla_{\mathcal{T}} g(x) \lambda^{\mathcal{T}}(dx).$$

Proposition. $(\mathcal{E}^{\mathcal{T}}, \mathcal{F}^{\mathcal{T}})$ is a local, regular Dirichlet form on $L^2(\mu^{\mathcal{T}})$.

NB. By saying the Dirichlet form is **local**, it is meant that

$$\mathcal{E}^{\mathcal{T}}(f,g) = 0$$

whenever the support of f and g are disjoint.

BROWNIAN MOTION ON REAL TREES

Let $(\mathcal{T}, d_{\mathcal{T}}, \rho_{\mathcal{T}})$ be a compact, rooted real tree, and $\mu^{\mathcal{T}}$ a finite Borel measure on $\mathcal{T}$ with full support.

From the standard theory above, there is a non-positive self-adjoint operator $\Delta_{\mathcal{T}}$ on $L^2(\mu^{\mathcal{T}})$ with $\mathcal{D}(\Delta_{\mathcal{T}}) \subseteq \mathcal{F}^{\mathcal{T}}$ and

$$\mathcal{E}^{\mathcal{T}}(f,g) = -\int_{\mathcal{T}} (\Delta_{\mathcal{T}} f)(x) g(x) \mu^{\mathcal{T}}(dx),$$

for every $f \in \mathcal{D}(\Delta_{\mathcal{T}})$, $g \in \mathcal{F}^{\mathcal{T}}$.

We define **Brownian motion** on $(\mathcal{T}, d_{\mathcal{T}}, \mu^{\mathcal{T}})$ to be the Markov process

$$\left(\left(X_t^{\mathcal{T}}\right)_{t \geq 0}, \left(P_x^{\mathcal{T}}\right)_{x \in \mathcal{T}}\right)$$

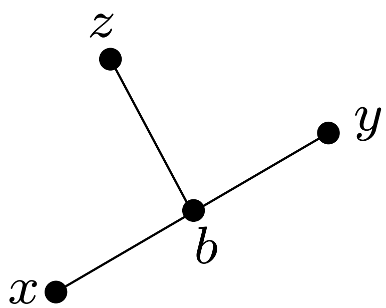
with semigroup $P_t^{\mathcal{T}} = e^{t\Delta_{\mathcal{T}}}$. Since $(\mathcal{E}^{\mathcal{T}}, \mathcal{F}^{\mathcal{T}})$ is local and regular, this is a diffusion.

PROPERTIES OF LIMITING PROCESS

Point recurrence: For $x, y \in \mathcal{T}$, $P_x^{\mathcal{T}}(\tau_y < \infty) = 1$.

Hitting probabilities: For $x, y, z \in \mathcal{T}$,

$$P_z^{\mathcal{T}}(\tau_x < \tau_y) = \frac{d_{\mathcal{T}}(b_{\mathcal{T}}(x,y,z), y)}{d_{\mathcal{T}}(x,y)}.$$



Occupation density: For $x, y \in \mathcal{T}$,

$$E_x^{\mathcal{T}} \int_0^{\tau_y} f(X_s^{\mathcal{T}}) ds = \int_{\mathcal{T}} f(x) d_{\mathcal{T}}(b_{\mathcal{T}}(x,y,z), y) \mu^{\mathcal{T}}(dz).$$

[cf. Aldous 1991]

RESISTANCE CHARACTERISATION: GRAPHS

As above, let $G = (V(G), E(G))$ be a finite graph, with edge weights $\lambda^G = (\lambda^G_e)_{e\in E(G)}$.

Suppose we view G as an electrical network with edges assigned conductances according to λ^G. Then the electrical resistance between x and y is given by

$$R_G(x,y)^{-1} = \inf\left\{\mathcal{E}^G(f,f):\ f(x)=1, f(y)=0\right\}.$$

R_G is a metric on $V(G)$, e.g. [Tetali 1991], and characterises the weights (and therefore the Dirichlet form) uniquely [Kigami 1995].

For a graph tree T, one has

$$R_T(x,y) = d_T(x,y),$$

where d_T is the weighted shortest path metric, with edges weighted according to $(1/\lambda^G_e)_{e\in E(G)}$.

RESISTANCE CHARACTERISATION: REAL TREES

Again, let $(\mathcal{T}, d_{\mathcal{T}}, \rho_{\mathcal{T}})$ be a compact, rooted real tree, and $\mu^{\mathcal{T}}$ a finite Borel measure on $\mathcal{T}$ with full support.

Similarly to the graph case, define the resistance on $\mathcal{T}$ by

$$R_{\mathcal{T}}(x,y)^{-1} = \inf\left\{\mathcal{E}^{\mathcal{T}}(f,f) :\ f \in \mathcal{F}^{\mathcal{T}},\ f(x)=1,\ f(y)=0\right\}.$$

One can check that $R_{\mathcal{T}} = d_{\mathcal{T}}$. By results of [Kigami 1995] on 'resistance forms', it is possible to check that this property characterises $(\mathcal{E}^{\mathcal{T}}, \mathcal{F}^{\mathcal{T}})$ uniquely amongst the collection of regular Dirichlet forms on $L^2(\mu^{\mathcal{T}})$.

Note that, for all $f \in \mathcal{F}_{\mathcal{T}}$,

$$|f(x)-f(y)|^2 \leq \mathcal{E}_{\mathcal{T}}(f,f)\, d_{\mathcal{T}}(x,y).$$

PROOF OF POINT RECURRENCE

[Fukushima/Oshima/Takeda 2011, Lemma 2.2.3] If ν is a positive Radon measure on $\mathcal{T}$ with finite energy integral, i.e.,

$$\left(\int_{\mathcal{T}}|f(x)|\nu(dx)\right)^2 \leq c\left(\mathcal{E}^{\mathcal{T}}(f,f)+\int_{\mathcal{T}}f(x)^2\mu^{\mathcal{T}}(dx)\right), \quad \forall f \in \mathcal{F}^{\mathcal{T}},$$

then ν charges no set of zero capacity.

Note that

$$\left(\int_{\mathcal{T}}|f(z)|\delta_x(dz)\right)^2 = f(x)^2 \leq 2(f(x)-f(y))^2 + 2f(y)^2.$$

Applying the resistance inequality to this bound, and integrating with respect to y yields

$$\left(\int_{\mathcal{T}}|f(y)|\delta_x(dy)\right)^2 \leq 2\,\mathrm{diam}\mathcal{T}_f\ \mathcal{E}^{\mathcal{T}}(f,f) + 2\int_{\mathcal{T}}f(y)^2\mu^{\mathcal{T}}(dy).$$

Thus points have strictly positive capacity.

PROOF OF OCCUPATION DENSITY FORMULA

Let $g(z) = g^y(x,z) = d_{\mathcal{T}}(b_{\mathcal{T}}(x,y,z),y)$, then

$$\nabla g = \mathbf{1}_{[[b_{\mathcal{T}}(\rho_{\mathcal{T}},x,y),x]]}(z) - \mathbf{1}_{[[b_{\mathcal{T}}(\rho_{\mathcal{T}},x,y),y]]}(z).$$

And for $h \in \mathcal{F}_{\mathcal{T}}$ with $h(y) = 0$,

$$\mathcal{E}_{\mathcal{T}}(g,h) = \int_{b_{\mathcal{T}}(\rho_{\mathcal{T}},x,y)}^{x} \nabla h(z)\lambda^{\mathcal{T}}(dz) - \int_{b_{\mathcal{T}}(\rho_{\mathcal{T}},x,y)}^{y} \nabla h(z)\lambda^{\mathcal{T}}(dz) = h(x).$$

Hence, if $Gf(x) := \int_{\mathcal{T}} g^y(x,z)f(z)\mu^{\mathcal{T}}(dz)$, then

$$\mathcal{E}_{\mathcal{T}}(Gf,h) = \int_{\mathcal{T}} f(z)h(z)\mu^{\mathcal{T}}(dz).$$

Since the resolvent is unique, to complete the proof it is enough to note that

$$\tilde{G}f(x) := E_x^{\mathcal{T}} \int_0^{\tau_y} f(X_s^{\mathcal{T}})ds = \int_0^{\infty} P_t^{\mathcal{T}\setminus\{y\}} f(x)dt$$

also satisfies the previous identity.

4. TRACES AND TIME CHANGE

TRACE OF THE DIRICHLET FORM

Through this section, let $(\mathcal{T}, d_{\mathcal{T}}, \rho_{\mathcal{T}})$ be a compact, rooted real tree, and $\mu^{\mathcal{T}}$ a finite Borel measure on $\mathcal{T}$ with full support.

Suppose $\mathcal{T}'$ is a non-empty subset of $\mathcal{T}$.

Define the trace of $(\mathcal{E}^{\mathcal{T}}, \mathcal{F}^{\mathcal{T}})$ on $\mathcal{T}'$ by setting:

$$\mathrm{Tr}\left(\mathcal{E}^{\mathcal{T}}|\mathcal{T}'\right)(g,g) := \inf\left\{\mathcal{E}^{\mathcal{T}}(f,f) :\ f \in \mathcal{F}^{\mathcal{T}},\ f|_{\mathcal{T}'} = g\right\},$$

where the domain of $\mathrm{Tr}(\mathcal{E}^{\mathcal{T}}|\mathcal{T}')$ is precisely the collection of functions for which the right-hand side is finite.

Theorem. If $\mathcal{T}'$ is closed, and $\mu^{\mathcal{T}'}$ is a finite Borel measure on $(\mathcal{T}', d_{\mathcal{T}})$ with full support, then $\mathrm{Tr}\left(\mathcal{E}^{\mathcal{T}}|\mathcal{T}'\right)$ is a regular Dirichlet form on $L^2(\mu^{\mathcal{T}'})$ [Fukushima/Oshima/Takeda 2011].

APPLICATION TO REAL TREES

Suppose $\mathcal{T}' \subseteq \mathcal{T}$ is closed and arcwise-connected (so that $(\mathcal{T}', d_{\mathcal{T}})$ is a real tree), equipped with a finite Borel measure $\mu^{\mathcal{T}'}$ of full support. We claim that

$$\mathcal{E}^{\mathcal{T}'} = \mathrm{Tr}\left(\mathcal{E}^{\mathcal{T}} | \mathcal{T}'\right).$$

Indeed, both are regular Dirichlet forms on $L^2(\mu^{\mathcal{T}'})$, and

$$\begin{aligned} &\inf\left\{\mathrm{Tr}\left(\mathcal{E}^{\mathcal{T}}|\mathcal{T}'\right)(g,g) :\ g(x)=1,\ g(y)=0\right\} \\ &= \inf\left\{\inf\left\{\mathcal{E}^{\mathcal{T}}(f,f) :\ f \in \mathcal{F}^{\mathcal{T}},\ f|_{\mathcal{T}'} = g\right\} :\ g(x)=1,\ g(y)=0\right\} \\ &= \inf\left\{\mathcal{E}^{\mathcal{T}}(f,f) :\ f \in \mathcal{F}^{\mathcal{T}},\ f(x)=1,\ f(y)=0\right\} \\ &= d_{\mathcal{T}}(x,y)^{-1}. \end{aligned}$$

In particular, $\mathrm{Tr}(\mathcal{E}^{\mathcal{T}}|\mathcal{T}')$ is the form naturally associated with Brownian motion on $(\mathcal{T}', d_{\mathcal{T}}, \mu^{\mathcal{T}'})$.

TIME CHANGE

Given a finite Borel measure ν with support $\mathcal{S} \subseteq \mathcal{T}$, let $(A_t)_{t\geq 0}$ be the positive continuous additive functional with Revuz measure ν. For example, if $X^{\mathcal{T}}$ admits jointly continuous local times $(L_t(x))_{x\in\mathcal{T},t\geq 0}$, i.e.

$$\int_0^t f(X_s^{\mathcal{T}})ds = \int_{\mathcal{T}} f(x)L_t(x)\mu_{\mathcal{T}}(dx), \qquad \forall f \in C(\mathcal{T}),$$

then

$$A_t = \int_{\mathcal{S}} L_t(x)\nu(dx).$$

Set

$$\tau(t) := \inf\{s > 0 : \ A_s > t\}.$$

Then $(X^{\mathcal{T}}_{\tau(t)})_{t\geq 0}$ is the Markov process naturally associated with $\mathrm{Tr}\left(\mathcal{E}^{\mathcal{T}}|\mathcal{S}\right)$, considered as a regular Dirichlet form on $L^2(\nu)$.

APPLICATION TO FINITE SUBSETS

Let V be a fine finite set of $\mathcal{T}$. If we define $\mathcal{E}^V = \mathrm{Tr}(\mathcal{E}^{\mathcal{T}}|V)$, then one can check for any finite measure μ^V on V with full support

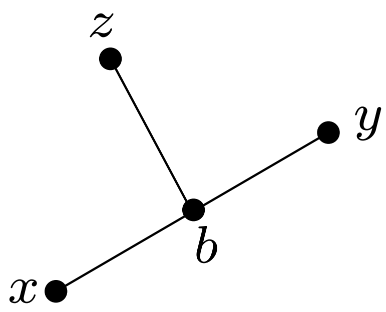
$$\begin{aligned}\mathcal{E}^V(f,g) &= \frac{1}{2}\sum_{x,y:x\sim y}\frac{1}{d_{\mathcal{T}}(x,y)}\left(f(x)-f(y)\right)\left(g(x)-g(y)\right)\\ &= -\sum_x(\Delta f)(x)g(x)\mu^V(\{x\}),\end{aligned}$$

where

$$\Delta f(x) := \sum_{y:y\sim x}\frac{1}{\mu^V(\{x\})d_{\mathcal{T}}(x,y)}\left(f(y)-f(x)\right).$$

PROOF OF HITTING PROBABILITIES FORMULA

Let $V = \{x, y, z, b_{\mathcal{T}}(x, y, z)\}$.



For any μ^V such that $\mu(\{v\}) \in (0, \infty)$ for all $v \in V$, we have $P_x^{\mathcal{T}}$-a.s.,

$$A_t = \int_0^t \mathbf{1}_V(X_s^{\mathcal{T}}) dA_s, \qquad \inf\{t : A_t > 0\} = \inf\{t : X_t^{\mathcal{T}} \in V\}.$$

[Fukushima/Oshima/Takeda 2011] It follows that the hitting distributions of $X_t^V = X_{\tau(t)}^{\mathcal{T}}$ and $X^{\mathcal{T}}$ are the same. Thus

$$P_z^{\mathcal{T}}(\tau_x < \tau_y) = P_z^V(\tau_x < \tau_y) = \frac{d_{\mathcal{T}}(b_{\mathcal{T}}(x, y, z), y)}{d_{\mathcal{T}}(x, y)}.$$

5. SCALING RANDOM WALKS ON GRAPH TREES

AIM

Let $(T_n)_{n\geq 1}$ be a sequence of finite graph trees, and μ_{T_n} the counting measure on $V(T_n)$.

(A1) There exist null sequences $(a_n)_{n\geq 1}$, $(b_n)_{n\geq 1}$ such that

$$\left(T_n, a_n d_{T_n}, b_n \mu_{T_n}, \rho_{T_n}\right) \to (\mathcal{T}, d_{\mathcal{T}}, \mu_{\mathcal{T}}, \rho_{\mathcal{T}})$$

with respect to the pointed Gromov-Hausdorff-Prohorov topology.

We aim to show that the corresponding simple random walks X^{T_n}, started from ρ_{T_n}, converge to Brownian motion $X^{\mathcal{T}}$ on $(\mathcal{T}, d_{\mathcal{T}}, \mu_{\mathcal{T}})$, started from $\rho_{\mathcal{T}}$.

ASSUMPTION ON LIMIT

From the convergence assumption (A1) we have that: $(\mathcal{T}, d_{\mathcal{T}}, \mu_{\mathcal{T}}, \rho_{\mathcal{T}})$ is a compact real tree, equipped with a finite Borel measure $\mu^{\mathcal{T}}$, and distinguished point $\rho_{\mathcal{T}}$.

(A2) There exists a constant $c > 0$ such that

$$\liminf_{r \to 0} \inf_{x \in \mathcal{T}} r^{-c} \mu_{\mathcal{T}} (B_{\mathcal{T}}(x, r)) > 0.$$

This property is not necessary, but allows a sample path proof.

In particular, it ensures that $X^{\mathcal{T}}$ admits jointly continuous local times $(L_t(x))_{x \in \mathcal{T}, t \geq 0}$, i.e.

$$\int_0^t f(X_s^{\mathcal{T}}) ds = \int_{\mathcal{T}} f(x) L_t(x) \mu_{\mathcal{T}}(dx), \qquad \forall f \in C(\mathcal{T}).$$

A NOTE ON THE TOPOLOGY

The assumption (A1) is equivalent to there existing isometric embeddings of $(T_n, d_{T_n})_{n\geq 1}$ and $(\mathcal{T}, a_n d_{\mathcal{T}})$ into the same metric space (M, d_M) such that:

$$d_M(\rho_{T_n}, \rho_{\mathcal{T}}) \to 0, \qquad d_M^H(T_n, \mathcal{T}) \to 0, \qquad d_M^P(b_n \mu_{T_n}, \mu_{\mathcal{T}}) \to 0.$$

Indeed, one can take

$$M = T_1 \sqcup T_2 \sqcup \cdots \sqcup \mathcal{T}$$

equipped with suitable metric (cf. end of Section 2).

We will identify the various objects with their embeddings into M, and show convergence of processes in the space $D(\mathbb{R}_+, M)$.

PROJECTIONS

Let $(x_i)_{i\geq 1}$ be a dense sequence in $\mathcal{T}$, and set

$$\mathcal{T}(k) := \cup_{i=1}^{k}[[\rho_{\mathcal{T}}, x_i]],$$

where $[[\rho_{\mathcal{T}}, x_i]]$ is the unique path from $\rho_{\mathcal{T}}$ to x_i in $\mathcal{T}$.

Let $\phi_k : \mathcal{T} \to \mathcal{T}(k)$ be the map such that $\phi_k(x)$ is the nearest point of $\mathcal{T}(k)$ to x. (We call this the **projection** of $\mathcal{T}$ onto $\mathcal{T}(k)$.)

For each n, choose $(x_i^n)_{i\geq 1}$ in T_n such that

$$d_M(x_i^n, x_i) \to 0,$$

and define the subtree $T_n(k)$ and projection $\phi_{n,k} : T_n \to T_n(k)$ similarly to above.

CONVERGENCE CRITERIA

It is possible to check that the assumption (A1) is equivalent to the following two conditions holding:

1. Convergence of finite dimensional distributions: for each k,

$$d_M^H(T_n(k), \mathcal{T}(k)) \to 0, \qquad d_M^P(b_n\mu_{n,k}, \mu_k) \to 0,$$

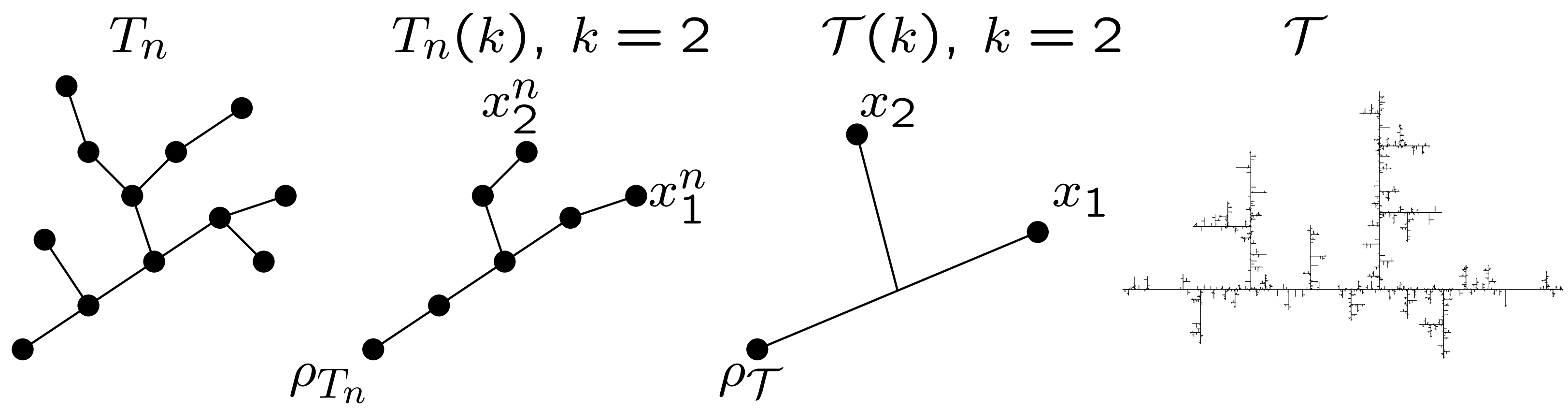
where $\mu_{n,k} := \mu_{T_n} \circ \phi_{n,k}^{-1}$ and $\mu_k := \mu_{\mathcal{T}} \circ \phi_k^{-1}$.

2. Tightness:

$$\lim_{k\to\infty} \limsup_{n\to\infty} d_M^H(T_n(k), T_n) = 0.$$

STRATEGY

Select $T_n(k)$ and $\mathcal{T}(k)$ as above:



Step 1: Show Brownian motion $X^{\mathcal{T}(k)}$ on $(\mathcal{T}(k), d_{\mathcal{T}}, \mu_k)$ converges to $X^{\mathcal{T}}$.
Step 2: For each k, construct processes $X^{T_n(k)}$ on graph subtrees that converge to $X^{\mathcal{T}(k)}$.
Step 3: Show $X^{T_n(k)}$ are close to X^{T_n} as $k \to \infty$.

STEP 1
APPROXIMATION OF LIMITING DIFFUSION

TIME CHANGE CONSTRUCTION

Define

$$A_t^k := \int_{\mathcal{T}} L_t(x)\mu_k(dx),$$

set

$$\tau_k(t) = \text{inf}\{s : \ A_s^k > t\}.$$

Then, we recall from Section 4, $X^{\mathcal{T}}_{\tau_k(t)}$ is the Markov process naturally associated with

$$\text{Tr}\left(\mathcal{E}^{\mathcal{T}}|\mathcal{T}(k)\right),$$

(note that supp$\mu_k = \mathcal{T}(k)$), considered as a Dirichlet form on $L^2(\mu_k)$.

Recall also that the latter process is Brownian motion $X^{\mathcal{T}(k)}$ on $(\mathcal{T}(k), d_{\mathcal{T}}, \mu_k)$.

CONVERGENCE OF DIFFUSIONS

By construction

$$d_M^P(\mu_k, \mu_{\mathcal{T}}) \leq \sup_{x \in \mathcal{T}} d_M(\phi_k(x), x) = d_M^H(\mathcal{T}(k), \mathcal{T}) \to 0.$$

Hence, applying the continuity of local times:

$$A_t^k = \int_{\mathcal{T}} L_t(x) \mu_k(dx) \to \int_{\mathcal{T}} L_t(x) \mu_{\mathcal{T}}(dx) = t,$$

uniformly over compact intervals.

Thus, we also have that $\tau_k(t) \to t$ uniformly on compacts. And, by continuity,

$$X_t^{\mathcal{T}(k)} = X_{\tau_k(t)}^{\mathcal{T}} \to X_t^{\mathcal{T}},$$

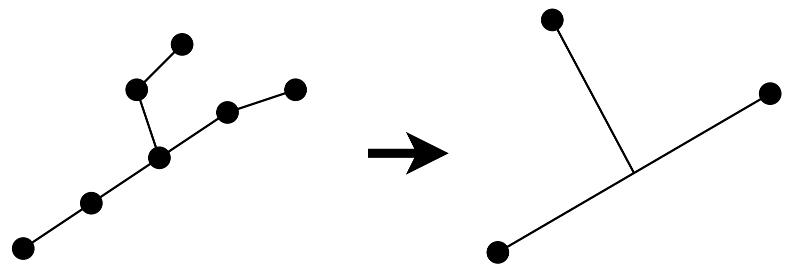
uniformly on compacts.

STEP 2
CONVERGENCE OF WALKS ON FINITE TREES

CONVERGENCE OF WALKS ON FINITE TREES EQUIPPED WITH LENGTH MEASURE

For fixed k,

$$T_n(k) \to \mathcal{T}(k).$$



If $J^{n,k}$ is the simple random walk on $T_n(k)$, then

$$\left(J^{n,k}_{tE_{n,k}/a_n}\right)_{t\geq 0} \to \left(X_t^{\mathcal{T}(k),\lambda_k}\right)_{t\geq 0},$$

where $E_{n,k} := \#E(T_n(k))$ and $X^{\mathcal{T}(k),\lambda_k}$ is the Brownian motion on $(\mathcal{T}(k), d_{\mathcal{T}}, \lambda_k)$, for λ_k equal to the length measure on $\mathcal{T}(k)$, normalised such that $\lambda_k(\mathcal{T}(k)) = 1$.

TIME CHANGE FOR LIMIT

For $(L_t^k(x))_{x\in\mathcal{T}(k),t\geq 0}$ the local times of $X^{\mathcal{T}(k),\lambda_k}$, write

$$\hat{A}_t^k := \int_{\mathcal{T}(k)} L_t^k(x)\mu_k(dx),$$

and set

$$\hat{\tau}_k(t) = \inf\{s : \hat{A}_s^k > t\}.$$

Then

$$\left(X_{\hat{\tau}_k(t)}^{\mathcal{T}(k),\lambda_k}\right)_{t\geq 0} = \left(X_t^{\mathcal{T}(k)}\right)_{t\geq 0}.$$

TIME CHANGE FOR GRAPHS

Let

$$\widehat{A}_m^{n,k} := \sum_{l=0}^{m-1} \frac{2\mu_{n,k}(\{J_l^{n,k}\})}{\deg_{n,k}(J_l^{n,k})} = \sum_{x \in T_n(k)} L_m^{n,k}(x)\mu_{n,k}(\{x\}),$$

where

$$L_m^{n,k}(x) := \frac{2}{\deg_{n,k}(x)} \sum_{l=0}^{m-1} \mathbf{1}_{\{J_l^{n,k}=x\}}.$$

If

$$\widehat{\tau}^{n,k}(m) := \max\{l :\ \widehat{A}_l^{n,k} \leq m\},$$

then

$$X_m^{T_n(k)} = J_{\widehat{\tau}^{n,k}(m)}^{n,k}$$

is the process with the same jump chain as $J^{n,k}$, and holding times given by $2\mu_{n,k}(\{x\})/\deg_{n,k}(x)$.

CONVERGENCE OF TIME-CHANGED PROCESSES

We have that

$$\left(a_n L^{n,k}_{tE_{n,k}/a_n}(x)\right)_{x\in T_n(k),t\geq 0} \to \left(L^k_t(x)\right)_{x\in\mathcal{T}(k),t\geq 0}, \qquad b_n\mu_{n,k}\to\mu_k.$$

This implies which implies

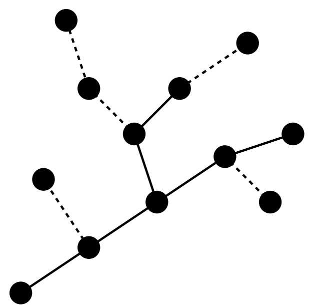
$$\begin{aligned} a_n b_n \widehat{A}^{n,k}_{tE_{n,k}/a_n} &= a_n b_n \int_{T_n(k)} L^{n,k}_{tE_{n,k}/a_n}(x)\mu_{n,k}(dx) \\ &\to \int_{\mathcal{T}(k)} L^k_t(x)\mu_k(dx) \\ &= \widehat{A}^k_t. \end{aligned}$$

Taking inverses and composing with $J^{n,k}$ and $X^{\mathcal{T}(k),\lambda_k}$ yields

$$X^{T_n(k)}_{t/a_nb_n} = J^{n,k}_{\widehat{\tau}^{n,k}(t/a_nb_n)} \to X^{\mathcal{T}(k),\lambda_k}_{\widehat{\tau}_k(t)} = X^{\mathcal{T}(k)}_t.$$

STEP 3
APPROXIMATING RANDOM WALKS ON WHOLE TREES

PROJECTION OF RANDOM WALK



$\phi_{n,k}$ is natural projection from T_n to $T_n(k)$.

Clearly

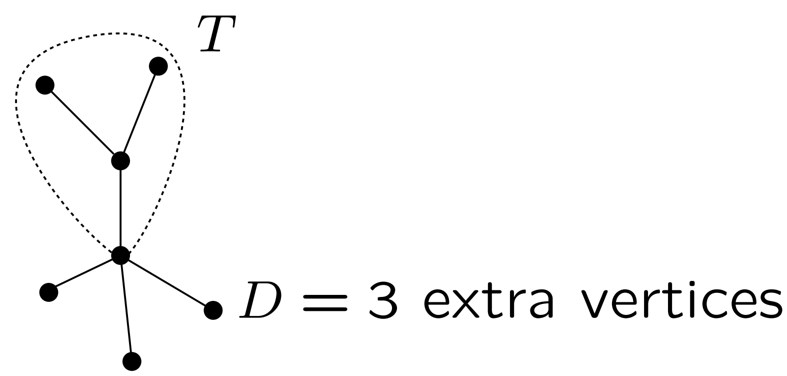
$$\begin{aligned}
&\lim_{k\to\infty}\limsup_{n\to\infty}\sup_{t\in[0,T]} d_M\left(X^{T_n}_{t/a_nb_n}, \phi_{n,k}(X^{T_n}_{t/a_nb_n})\right)\\
&\leq \lim_{k\to\infty}\limsup_{n\to\infty}\sup_{x\in V(T_n)} d_M(x,\phi_{n,k}(x))\\
&= \lim_{k\to\infty}\limsup_{n\to\infty} d_M^H(T_n(k),T_n)\\
&= 0.
\end{aligned}$$

Moreover, can couple projected process $\phi_{n,k}(X^{T_n})$ and time-changed process $X^{T_n(k)}$ to have same jump chain $J^{n,k}$. Recall $X^{T_n(k)}$ waits at a vertex x a fixed time $2\mu_{n,k}(\{x\})/\mathrm{deg}_{n,k}(x)$.

ELEMENTARY SIMPLE RANDOM WALK IDENTITY

Let T be a rooted graph tree, and attach D extra vertices at its root, each by a single edge.

e.g.



If $\alpha(T, D)$ is the expected time for a simple random walk started from the root to hit one of the extra vertices, then

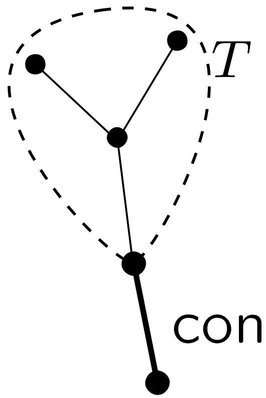
$$\alpha(T, D) = \frac{2\#V(T) - 2 + D}{D}.$$

In particular, if $D = 2$, then

$$\alpha(T, D) = \#V(T).$$

PROOF

We consider modified graph $G = T \cup \{\rho\}$ obtained by identifying extra vertices into one vertex:



If τ_ρ^+ is the return time to ρ, then

$$\alpha(T,D) + 1 = E_\rho^G \tau_\rho^+ = \frac{1}{\pi(\rho)},$$

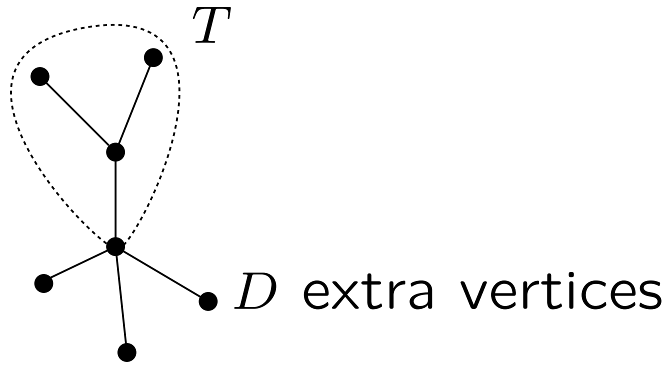
where π is the invariant probability measure of the random walk. In particular, writing $\lambda(v) = \sum_{e:\, v \in e} \lambda_e$,

$$\pi(\rho) = \frac{\lambda(\rho)}{\sum_v \lambda(v)} = \frac{D}{2(D + \#E(T))} = \frac{D}{2(D + \#V(T) - 1)}.$$

SECOND MOMENT ESTIMATE

Again, let T be a rooted graph tree, and attach D extra vertices at its root, each by a single edge.

e.g.



If $\beta(T, D)$ is the second moment of the time for a simple random walk started from the root to hit one of the extra vertices, then there exists a universal constant c such that

$$\beta(T, D) \leq c\left(\#V(T)^2 \times (1 + h(T)) + Dh(T)\right),$$

where $h(T)$ is the height of T.

PROOF

Let $G = T \cup \{\rho\}$ be the modified graph as in the previous proof. If $\lambda(G) = \sum_v \lambda(v) = 2\sum_e \lambda_e$ and $r(G) = \max_{x,y\in G} R(x,y)$, then we claim

$$P_\rho^G\left(\tau_\rho^+ \geq a\right) \leq \frac{c_1}{r(G)D} e^{-c_2 a/\lambda(G)r(G)}.$$

Indeed, applying the Markov property repeatedly, we obtain

$$P_\rho^G\left(\tau_\rho^+ \geq a\right) \leq P_\rho^G\left(\tau_\rho^+ \geq a/k\right)\left(\max_{x\in V(T)} P_x^G\left(\tau_\rho \geq a/k\right)\right)^{k-1}.$$

For $k = a/2\lambda(G)r(G)$, we have

$$P_\rho^G\left(\tau_\rho^+ \geq a/k\right) \leq \frac{kE_\rho^G \tau_\rho^+}{a} = \frac{1}{2r(G)D},$$

and also, by the commute time identity,

$$\max_{x\in V(T)} P_x^G\left(\tau_x \geq a/k\right) \leq \max_{x\in V(T)} \frac{kE_x^G \tau_\rho}{a} \leq \max_{x\in V(T)} \frac{kR(x,\rho)\lambda(G)}{a} \leq \frac{1}{2}.$$

PROOF (CONT.)

It follows that

$$E_\rho^G\left((\tau_\rho^+)^2\right) \leq \frac{c_3\lambda(G)^2 r(G)}{D}.$$

Since

$$\beta(T,D) = E_\rho^G\left((\tau_\rho^+ - 1)^2\right),$$

we can then use that

$$\lambda(G) = 2(D + \#V(T) - 1), \qquad r(G) \leq 2(h(T) + D^{-1})$$

to complete the proof.

CLOSENESS OF CLOCK PROCESSES

Suppose the mth jump of $\phi_{n,k}(X^{T_n})$ happens at $A_m^{n,k}$. Applying the above moment estimates and Kolmogorov's maximum estimate, i.e. if X_i are independent, centred, then

$$\mathbf{P}(\max_{l=1,\dots,m} |\sum_{i=1}^{l} X_i| \geq x) \leq x^{-2} \sum_{i=1}^{m} \mathbf{E} X_i^2,$$

we deduce

$$\mathbf{P}\left(\max_{m \leq t E_{n,k}/a_n} \left| A_m^{n,k} - \widehat{A}_m^{n,k} \right| \geq \varepsilon / a_n b_n \right) \to 0$$

in probability as n and then k diverge.

CONCLUSION

Let $(T_n)_{n\geq 1}$ be a sequence of finite graph trees.

Suppose that there exist null sequences $(a_n)_{n\geq 1}$, $(b_n)_{n\geq 1}$ such that

$$\left(T_n, a_n d_{T_n}, b_n \mu_{T_n}, \rho_{T_n}\right) \to (\mathcal{T}, d_{\mathcal{T}}, \mu_{\mathcal{T}}, \rho_{\mathcal{T}})$$

with respect to the pointed Gromov-Hausdorff-Prohorov topology, and $\mathcal{T}$ satisfies a polynomial lower volume bound.

It is then possible to isometrically embed $(T_n)_{n\geq 1}$ and $\mathcal{T}$ into the same metric space (M, d_M) such that

$$\left(a_n X^{T_n}_{t/a_n b_n}\right)_{t\geq 0} \to \left(X^{\mathcal{T}}_t\right)_{t\geq 0}$$

in distribution in $C(\mathbb{R}_+, M)$, where we assume $X^{T_n}_0 = \rho_{T_n}$ for each n, and also $X^{\mathcal{T}}_0 = \rho_{\mathcal{T}}$.

REMARKS

(i) Can extend to locally compact case.
(ii) Alternative proof given in [Athreya/ Löhr/Winter 2014] (in a slightly more general setting) under the weaker assumption: for each $\delta > 0$,

$$\liminf_{n\to\infty} \inf_{x\in T_n} \mu_{T_n}(B_{T_n}(\rho_{T_n}, \delta/a_n)) > 0.$$

(iii) Embeddings can be described measurably, and chosen so result applies to random trees to give convergence of annealed laws. In particular, if

$$\left(T_n, a_n d_{T_n}, b_n \mu_{T_n}, \rho_{T_n}\right) \to (\mathcal{T}, d_{\mathcal{T}}, \mu_{\mathcal{T}}, \rho_{\mathcal{T}})$$

in distribution, then for appropriate embeddings

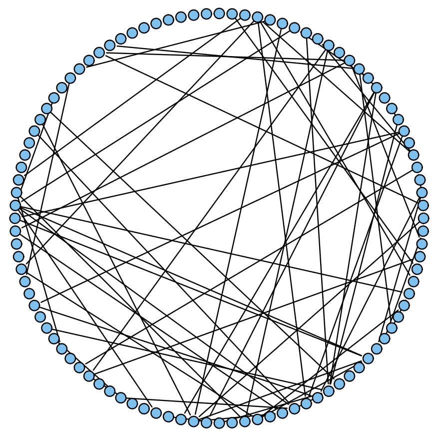
$$\int P^{T_n}_{\rho_{T_n}}((a_n X^{T_n}_{t/a_n b_n})_{t\geq 0} \in \cdot)\mathbb{P}(dT_n) \to \int P^{\mathcal{T}}_{\rho_{\mathcal{T}}}((X^{\mathcal{T}}_t)_{t\geq 0} \in \cdot)\mathbb{P}(d\mathcal{T}).$$

Applies to critical, finite variance GW trees conditioned on their size, with $a_n = n^{-1/2}$, $b_n = n^{-1}$.

6. FUSING AND THE CRITICAL RANDOM GRAPH

CRITICAL ERDŐS-RÉNYI RANDOM GRAPH (RECALLED)

$G(n,p)$ is obtained via bond percolation with parameter p on the complete graph with n vertices. We concentrate on critical window: $p = n^{-1} + \lambda n^{-4/3}$. e.g. $n = 100$, $p = 0.01$:



[Addario-Berry, Broutin, Goldschmidt] Considering the connected components as metric spaces,

$$(n^{-1/3}\mathcal{C}_1^n, n^{-1/3}\mathcal{C}_2^n, \dots) \to (\mathcal{M}_1, \mathcal{M}_2, \dots),$$

where $(\mathcal{M}_1, \mathcal{M}_2, \dots)$ is a sequence of random metric spaces.

CONDITIONING $\mathcal{C}_1^n$ ON ITS SIZE

For $m \in \mathbb{N}$, can construct $\mathcal{C}_1^n | \{\#\mathcal{C}_1^n = m\}$ as follows: first, choose an m-vertex random labelled tree T_m^p according to

$$\mathbf{P}(T_m^p = T) \propto (1-p)^{-a(T)},$$

where $a(T)$ is the number of extra edges 'permitted' by T. Then, add extra edges independently with probability p to form G_m^p.

If G is a connected graph with depth-first tree T and surplus s,

$$\begin{aligned} \mathbf{P}(G_m^p = G) &\propto (1-p)^{-a(T)} p^s (1-p)^{a(T)-s} = (p/(1-p))^s \\ &\propto p^{m-1+s}(1-p)^{\binom{m}{2}-m+1-s} = \mathbf{P}(G(m,p) = G). \end{aligned}$$

Finally, observe $\mathcal{C}_1^n | \{\#\mathcal{C}_1^n = m\} \sim G(m,p) | \{G(m,p) \text{ connected}\}$.

TILTING VIA THE EXCURSION AREA

In the discrete setting, the 'permitted' extra edges correspond to lattice points under the depth-first walk of the graph tree; the total number of them is (nearly) the area below this function.

In the continuous setting, an analogous construction of $\mathcal{M}_1$ is possible: first, choose a random excursion $\tilde{e}$ according to the tilted measure

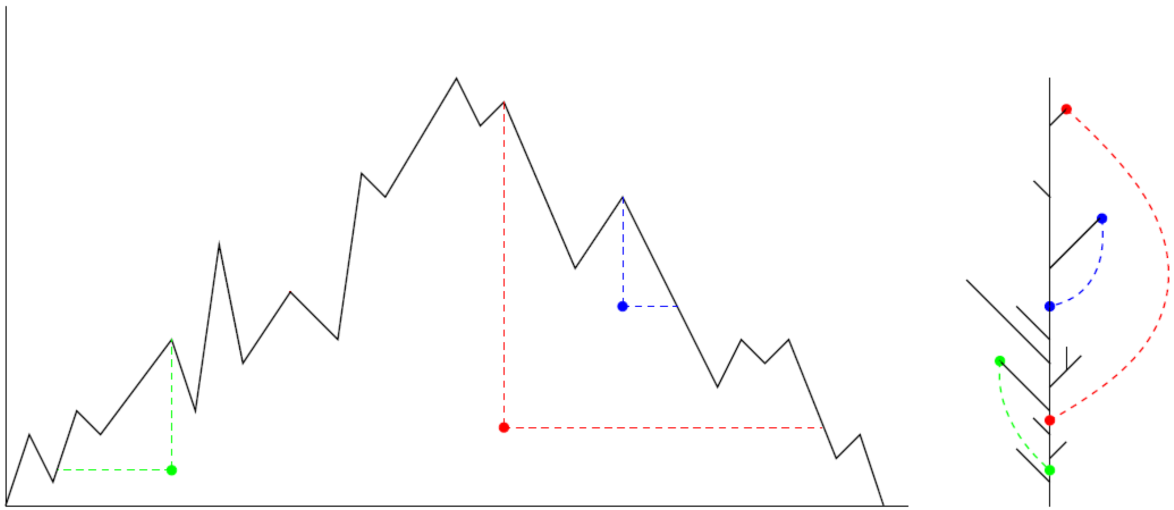
$$\mathbf{P}(\tilde{e} \in df) = \frac{\mathbf{P}(e \in df)\mathrm{exp}(\int_0^1 f(t)dt)}{\mathbf{E}(\mathrm{exp}(\int_0^1 \mathrm{e(t)dt}))},$$

where e is the normalised Brownian excursion.

Define $\tilde{\mathcal{T}} := \mathcal{T}_{\tilde{e}}$.

POINT PROCESS DESCRIBING CONNECTIONS

Let $\mathcal{P}$ be a unit intensity Poisson process on the plane. Points of $\mathcal{P}$ that lie below the excursion $\tilde{e}$ describe pairs of vertices to 'glue' together.



Picture produced by Christina Goldschmidt.

A point at (t, x) identifies the vertex v at height $\tilde{e}(t)$ with the vertex at distance x along the path from the root to v.

CRITICAL RANDOM GRAPH SCALING LIMIT [Addario-Berry, Broutin, Goldschmidt]

Up to a random scaling factor depending on λ, the random metric space scaling limit $(\mathcal{M}_1, d_{\mathcal{M}_1})$ of the largest component of the critical random graph is then defined as follows:

Let $\mathcal{M}_1$ be the image of the natural quotient map ϕ induced by the gluing of pairs of vertices of $\tilde{\mathcal{T}}$ according to $\mathcal{P}$.

Set $d_{\mathcal{M}_1}$ to be the quotient metric on $\mathcal{M}_1$, i.e.

$$d_{\mathcal{M}_1}(\bar{x}, \bar{y}) = \inf \left\{ \sum_{i=1}^{k} d_{\tilde{\mathcal{T}}}(x_i, y_i) : \bar{x}_1 = \bar{x}, \bar{y}_i = \bar{x}_{i+1}, \bar{y}_k = \bar{y} \right\},$$

where $\bar{x} := \phi(x)$.

FUSING THE DIRICHLET FORM ON $\mathcal{T}$

Recall $\mathcal{M}_1$ is obtained by gluing together a finite number of pairs of vertices of $\mathcal{T}$, and $\phi : \tilde{\mathcal{T}} \to \mathcal{M}_1$ is the natural quotient map.

Let $(\mathcal{E}_{\tilde{\mathcal{T}}}, \mathcal{F}_{\tilde{\mathcal{T}}})$ be the Dirichlet form on $(\tilde{\mathcal{T}}, d_{\tilde{\mathcal{T}}}, \mu^{\tilde{\mathcal{T}}})$.

Define a quadratic form on the glued space by setting

$$\mathcal{E}_{\mathcal{M}_1}(f,f) := \mathcal{E}_{\tilde{\mathcal{T}}}(f \circ \phi, f \circ \phi),$$

for any $f \in \mathcal{F}_{\mathcal{M}_1}$, where

$$\mathcal{F}_{\mathcal{M}_1} := \{f : \mathcal{M}_1 \to \mathbb{R} : \ f \circ \phi \in \mathcal{F}_{\tilde{\mathcal{T}}}\}.$$

$(\mathcal{E}_{\mathcal{M}_1}, \mathcal{F}_{\mathcal{M}_1})$ is a local, regular Dirichlet form on $L^2(\mathcal{M}_1, \mu^{\mathcal{M}_1})$, where $\mu^{\mathcal{M}_1} := \mu^{\tilde{\mathcal{T}}} \circ \phi^{-1}$. We call the corresponding Markov diffusion $X^{\mathcal{M}_1}$ Brownian motion on $\mathcal{M}_1$.

A FIRST EXAMPLE OF FUSING

For $(\mathcal{T}, d_{\mathcal{T}}, \mu^{\mathcal{T}}) = ([0,1], \text{Euclidean}, \text{Lebesgue})$,

$$\mathcal{E}^{\mathcal{T}}(f,f) = \int_{[0,1]} f'(x)^2 dx,$$

and $X^{\mathcal{T}}$ is reflected Brownian motion on $[0,1]$.

If 0 and 1 are 'fused', $(\mathcal{M}, d_{\mathcal{M}})$ is the circle of unit circumference equipped with its usual metric, $\mu^{\mathcal{M}}$ is the one-dimensional Hausdorff measure on this, and

$$\mathcal{E}^{\mathcal{M}}(f,f) = \int_{\mathcal{M}} f'(x)^2 dx.$$

(Note the integral is over the circle). The corresponding process $X^{\mathcal{M}}$ is Brownian motion on the circle.

SCALING LIMIT FOR RANDOM WALKS ON CRITICAL RANDOM GRAPHS

Essentially the same argument as for GW trees works:
- select subgraphs consisting of a finite number of line segments.
- prove convergence on these.
- show these are close to processes of interest.

Let $\mathcal{C}_1^n$ be the largest component of random graph in the critical window, $p = n^{-1} + \lambda n^{-4/3}$, then

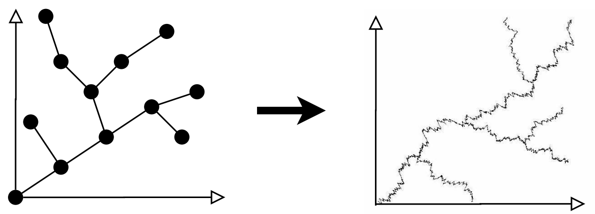
$$\left(n^{-1/3} X^{\mathcal{C}_1^n}_{\lfloor tn \rfloor}\right)_{t\geq 0} \to \left(X_t^{\mathcal{M}_1}\right)_{t\geq 0},$$

in distribution in both a quenched (for almost-every environment) and annealed (averaged over environments) sense.

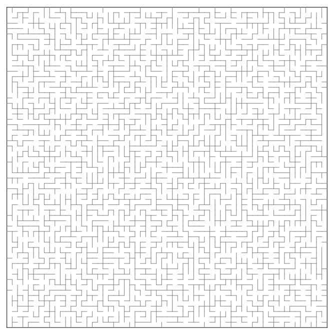
7. SPATIAL EMBEDDINGS

GRAPH TREES EMBEDDED IN EUCLIDEAN SPACE

Recall from the motivating examples, the branching random walk:



and the uniform spanning tree:



GH TOPOLOGY WITH SPATIAL EMBEDDING

Define $\mathbb{T}$ to be the collection of measured, rooted, spatial trees, i.e.

$$(\mathcal{T}, d_{\mathcal{T}}, \mu_{\mathcal{T}}, \phi_{\mathcal{T}}, \rho_{\mathcal{T}}),$$

where:

- $(\mathcal{T}, d_{\mathcal{T}})$ is a complete and locally compact real tree;
- $\mu_{\mathcal{T}}$ is a locally finite Borel measure on $(\mathcal{T}, d_{\mathcal{T}})$;
- $\phi_{\mathcal{T}}$ is a continuous map from $(\mathcal{T}, d_{\mathcal{T}})$ into $\mathbb{R}^d$;
- $\rho_{\mathcal{T}}$ is a distinguished vertex in $\mathcal{T}$.

On $\mathbb{T}_c$ (compact trees only), define a distance Δ_c by

$$\inf_{\substack{M,\psi,\psi',\mathcal{C}: \\ (\rho_{\mathcal{T}},\rho'_{\mathcal{T}})\in\mathcal{C}}} \left\{ d_M^P(\mu_{\mathcal{T}} \circ \psi^{-1}, \mu'_{\mathcal{T}} \circ \psi'^{-1}) + \sup_{(x,x')\in\mathcal{C}} \left(d_M(\psi(x), \psi'(x')) + \left|\phi_{\mathcal{T}}(x) - \phi'_{\mathcal{T}}(x')\right| \right) \right\}$$

Can be extended to locally compact case.

CONVERGENCE OF SRW

Let $(T_n)_{n\geq 1}$ be a sequence of finite graph trees, and X^{T_n} the SRW on T_n.

Suppose that there exist null sequences $(a_n)_{n\geq 1}$, $(b_n)_{n\geq 1}$, $(c_n)_{n\geq 1}$ such that

$$\left(T_n, a_n d_{T_n}, b_n \mu_{T_n}, c_n \phi_{T_n}, \rho_{T_n}\right) \rightarrow (\mathcal{T}, d_{\mathcal{T}}, \mu_{\mathcal{T}}, \phi_{\mathcal{T}}, \rho_{\mathcal{T}})$$

in $(\mathbb{T}_c, \Delta_c)$, where $(\mathcal{T}, d_{\mathcal{T}}, \mu_{\mathcal{T}}, \phi_{\mathcal{T}}, \rho_{\mathcal{T}})$ is an element of $\mathbb{T}_c^*$ – those for which a polynomial lower volume bound is satisfied. Let $X^{\mathcal{T}}$ be Brownian motion on $\mathcal{T}$, then

$$\left(c_n \phi_{T_n}\left(X^{T_n}_{t/a_n b_n}\right)\right)_{t\geq 0} \rightarrow \left(\phi_{\mathcal{T}}\left(X^{\mathcal{T}}_t\right)\right)_{t\geq 0}$$

in distribution in $C(\mathbb{R}_+, \mathbb{R}^d)$, where we assume $X_0^{T_n} = \rho_{T_n}$ for each n, and also $X_0^{\mathcal{T}} = \rho_{\mathcal{T}}$.

Again, can extend to locally compact case.

BRANCHING RANDOM WALK (RECALLED)

We will call a pair (T, ϕ), where T is a graph tree and $\phi : T \to \mathbb{R}^d$ a **graph spatial tree**.

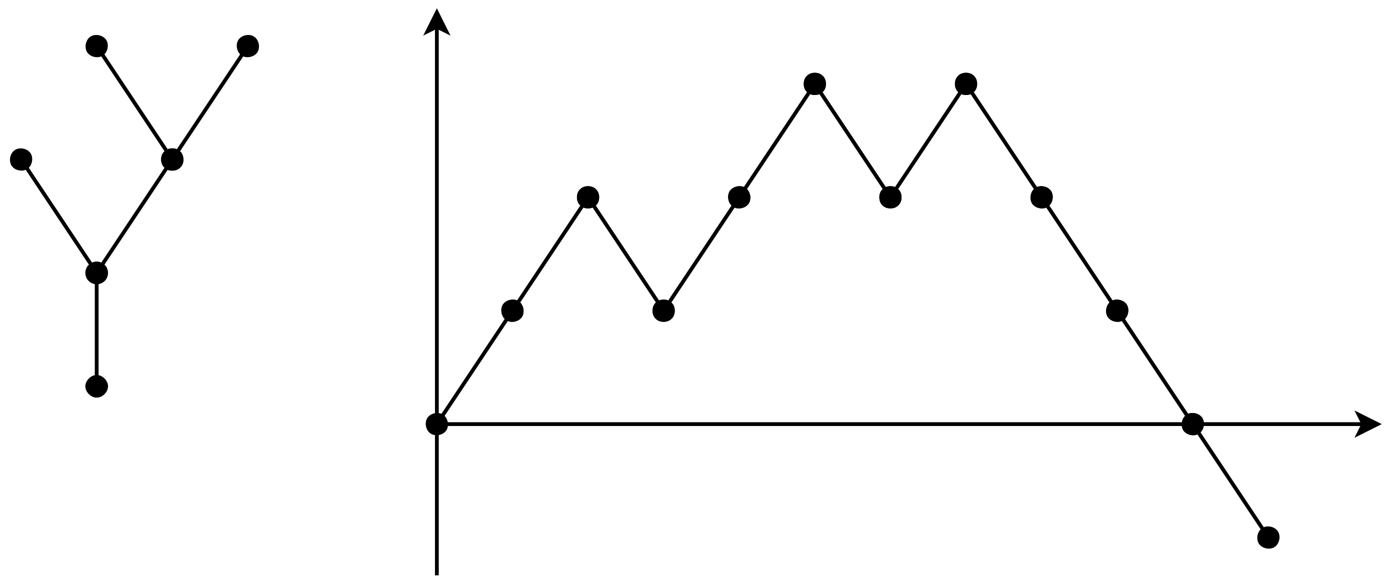
Given a graph tree T with root ρ, let $(\delta(e))_{e \in E(T)}$ be a collection of edge-indexed, i.i.d. random variables. Define $\phi : T \to \mathbb{R}^d$ by setting

$$\phi(v) = \sum_{e \in [[\rho, v]]} \delta(e);$$

note $(\phi(v))_{v \in T}$ is a **tree-indexed random walk**. In particular, if T is the tree generated by a branching process started with one initial ancestor, then the locations of $(\phi(v))_{v \in \text{generation } n}$, $n \geq 0$, form a **branching random walk**.

DISCRETE TOUR

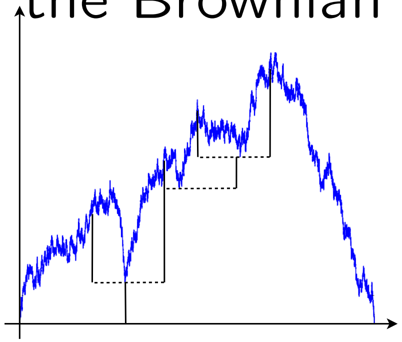
Given an ordered graph spatial tree (T, ϕ), recall its contour function $(C(t))_{t \in [0,2(n-1)]}$:



Let $(R(t))_{t \in [0,2(n-1)]}$ be defined by setting $R(t) = \phi([t])$, where $[t]_T \in T$ is the vertex in t visited by the contour process at time t. We call (C, R) the tour associated with (T, ϕ).

THE BROWNIAN TOUR

Consider a realisation of the Brownian excursion $(e(t))_{t\in[0,1]}$,



and its associated real tree $\mathcal{T}_e = [0,1]/\sim$, where $s \sim t$ iff $d_e(s,t) = 0$. Let $\phi : \mathcal{T}_e \to \mathbb{R}^d$ be a tree-indexed Brownian motion, i.e. $(\phi(v))_{v\in\mathcal{T}_e}$ is centred, Gaussian and

$$\mathrm{Cov}(\phi(v), \phi(v')) = d_{\mathcal{T}_e}(\rho_{\mathcal{T}_e}, b_{\mathcal{T}_e}(\rho_{\mathcal{T}_e}, v, v')).$$

Almost-surely when e is a Brownian excursion, this has a continuous version, see [Duquesne/Le Gall 2005].

Define $(r(t))_{t\in[0,1]} \in C([0,1], \mathbb{R}^d)$ by setting $r(t) = \phi([t])$. The process (e, r) is then the **Brownian tour**.

CONVERGENCE OF TOURS

Suppose T_n are critical Galton-Watson trees with finite exponential moment, aperiodic offspring distribution, and that $\delta(e)$ are centred and satisfy $\mathbb{P}(\delta(e) > x) = o(x^{-4})$. Let σ^2 be the variance of the offspring distribution, and $\mathrm{Var}\delta(e) = \Sigma$. Then

$$\left(n^{-1/2}C_{2(n-1)t}, n^{-1/4}R_{2(n-1)t}\right)_{t\in[0,1]} \to (\sigma_e e_t, \sigma_r r_t)_{t\in[0,1]},$$

in distribution in $C([0,1], \mathbb{R}_+ \times \mathbb{R}^d)$, where

$$\sigma_e = \frac{2}{\sigma}, \qquad \sigma_r = \Sigma\sqrt{\frac{2}{\sigma}}$$

[Janson/Marckert 2005].

SRW ON BRW CONVERGENCE

From the previous result, we deduce similarly to Section 2, that

$$\left(T_n, n^{-1/2} d_{T_n}, n^{-1} \mu_{T_n}, n^{-1/4} \phi_{T_n}, \rho_{T_n}\right) \to (\mathcal{T}, d_{\mathcal{T}}, \mu_{\mathcal{T}}, \phi_{\mathcal{T}}, \rho_{\mathcal{T}})$$

in $(\mathbb{T}_c, \Delta_c)$, where $(\mathcal{T}, d_{\mathcal{T}}, \mu_{\mathcal{T}}, \phi_{\mathcal{T}}, \rho_{\mathcal{T}})$ is suitably rescaled copy of the Brownian continuum random tree, embedded into $\mathbb{R}^d$ by a tree-indexed Brownian motion. Consequently, under the annealed law,

$$\left(n^{-1/4} \phi_{T_n}\left(X^{T_n}_{tn^{3/2}}\right)\right)_{t \geq 0} \to \left(\phi_{\mathcal{T}}\left(X^{\mathcal{T}}_t\right)\right)_{t \geq 0}$$

in distribution in $C(\mathbb{R}_+, \mathbb{R}^d)$, where we assume $X_0^{T_n} = \rho_{T_n}$ for each n, and also $X_0^{\mathcal{T}} = \rho_{\mathcal{T}}$.

Note in non-lattice case, this also implies convergence of walks on embedded graphs $G_n = \phi_{T_n}(T_n)$.

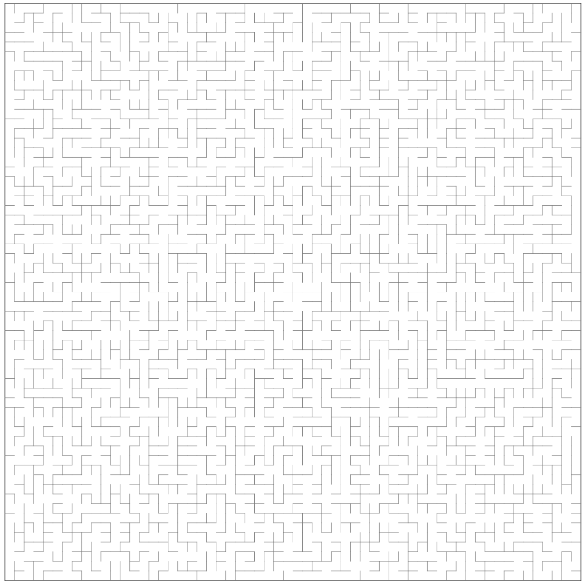
SOME OPEN QUESTIONS

At least in $d \geq 8$, where $\phi_{\mathcal{T}}(\mathcal{T})$ is itself a tree, is $\phi_{\mathcal{T}}(X^{\mathcal{T}})$ the scaling limit of random walk on lattice branching random walk?

How about for a large critical percolation cluster?

(The natural conjecture is yes!)

TWO-DIMENSIONAL UNIFORM SPANNING TREE (RECALLED)

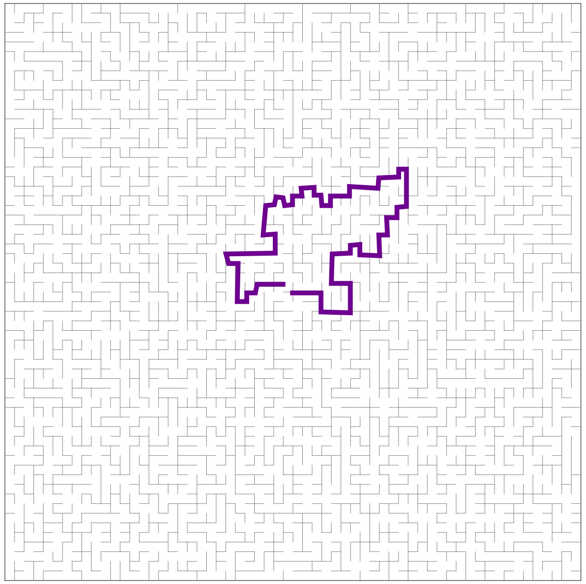


Let $\Lambda_n := [-n, n]^2 \cap \mathbb{Z}^2$.

Let $\mathcal{U}^{(n)}$ be a spanning tree of Λ_n selected uniformly at random from all possibilities.

The UST on $\mathbb{Z}^2$, $\mathcal{U}$, is then the local limit of $\mathcal{U}^{(n)}$.

TWO-DIMENSIONAL UNIFORM SPANNING TREE (RECALLED)



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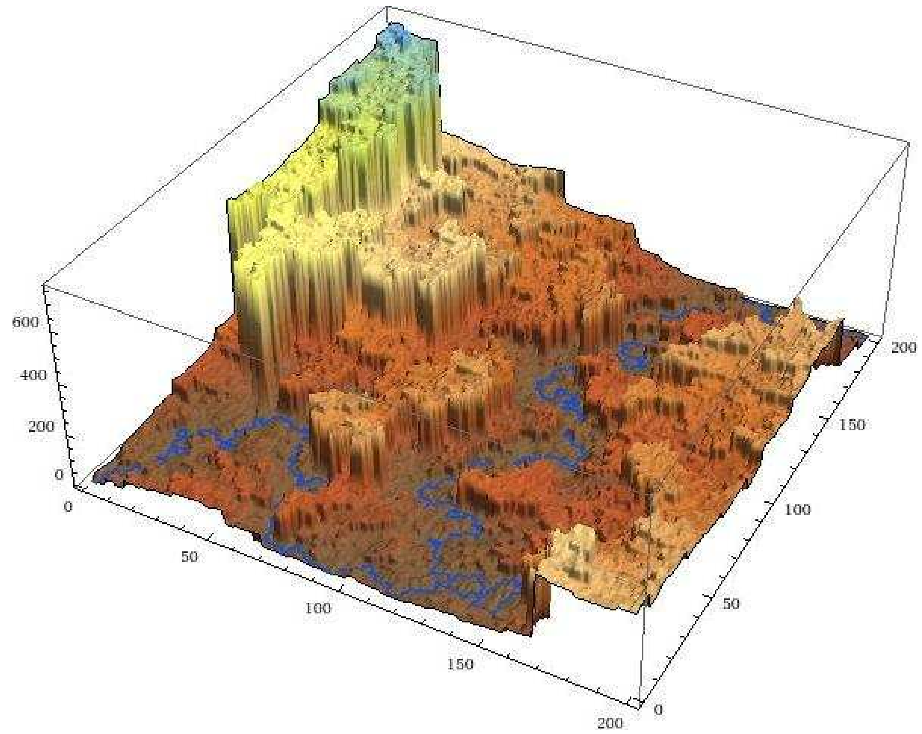


The distances in the tree to the path between opposite corners in a uniform spanning tree in a 200×200 grid.

Picture: Lyons/Peres: Probability on trees and networks

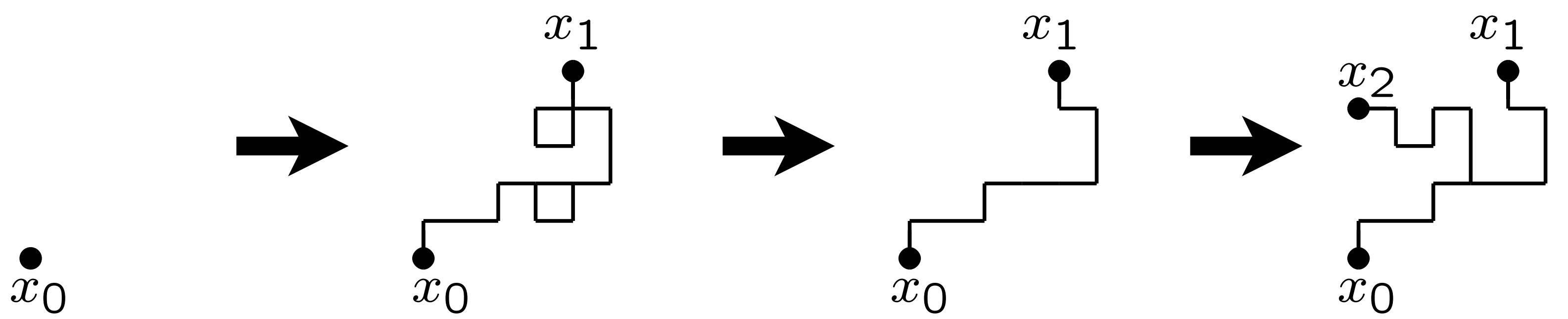
WILSON'S ALGORITHM ON $\mathbb{Z}^2$

Let $x_0 = 0, x_1, x_2, \ldots$ be an enumeration of $\mathbb{Z}^2$.

Let $\mathcal{U}(0)$ be the graph tree consisting of the single vertex x_0.

Given $\mathcal{U}(k-1)$ for some $k \geq 1$, define $\mathcal{U}(k)$ to be the union of $\mathcal{U}(k-1)$ and the loop-erased random walk (LERW) path run from x_k to $\mathcal{U}(k-1)$.

The UST $\mathcal{U}$ is then the local limit of $\mathcal{U}(k)$.

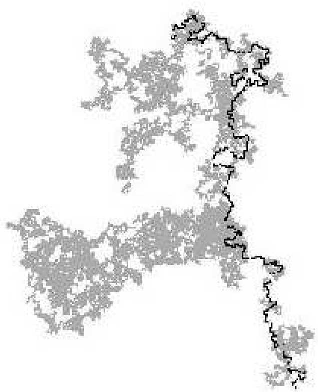


LERW SCALING IN $\mathbb{Z}^d$

Consider LERW as a process $(L_n)_{n\geq 0}$ (assume original random walk is transient).

In $\mathbb{Z}^d$, $d \geq 5$, L rescales diffusively to Brownian motion [Lawler].

In $\mathbb{Z}^4$, with logarithmic corrections rescales to Brownian motion [Lawler].

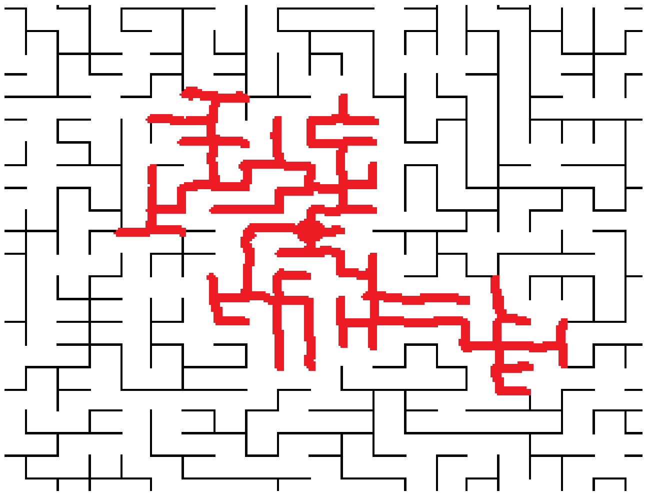


Picture: Ariel Yadin

In $\mathbb{Z}^3$, $\{L_n : n \in [0, \tau]\}$ has a scaling limit [Kozma].

In $\mathbb{Z}^2$, $\{L_n : n \in [0, \tau]\}$ has SLE(2) scaling limit [Lawler/Schramm/Werner]. Growth exponent is 5/4 [Kenyon, Masson, Lawler].

VOLUME ESTIMATES [BARLOW/MASSON 2011]



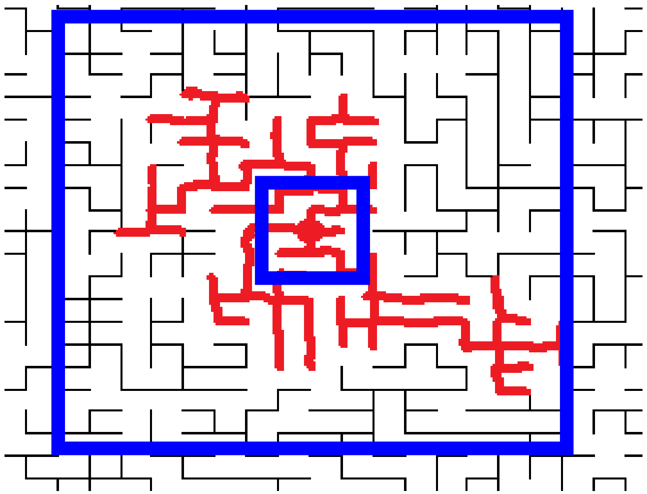
With high probability,

$$B_E(x, \lambda^{-1}R) \subseteq B_{\mathcal{U}}(x, R^{5/4}) \subseteq B_E(x, \lambda R),$$

as $R \to \infty$ then $\lambda \to \infty$. It follows that with high probability,

$$\mu_{\mathcal{U}}\left(B_{\mathcal{U}}(x, R)\right) \asymp R^{8/5}.$$

VOLUME ESTIMATES [BARLOW/MASSON 2011]



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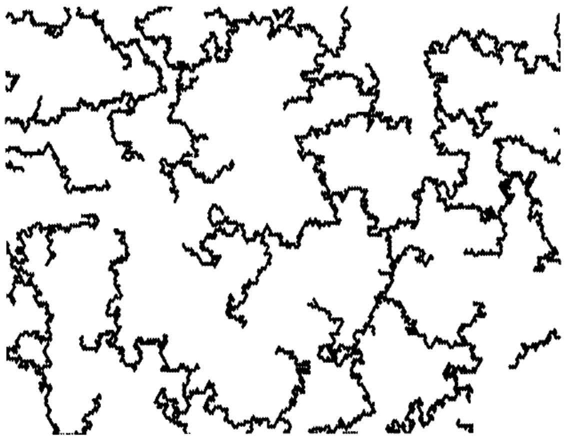
$$\mu_{\mathcal{U}}\left(B_{\mathcal{U}}(x, R)\right) \asymp R^{8/5}.$$

UST SCALING [SCHRAMM 2000]

Consider $\mathcal{U}$ as an ensemble of paths:

$$\mathfrak{U} = \left\{ (a, b, \pi_{ab}) : \ a, b \in \mathbb{Z}^2 \right\},$$

where π_{ab} is the unique arc connecting a and b in $\mathcal{U}$, as an element of the compact space $\mathcal{H}(\dot{\mathbb{R}}^2 \times \dot{\mathbb{R}}^2 \times \mathcal{H}(\dot{\mathbb{R}}^2))$, cf. [Aizenman/Burchard/Newman/Wilson].



Picture: Oded Schramm

Scaling limit $\mathfrak{T}$ almost-surely satisfies:

- each pair $a, b \in \dot{\mathbb{R}}^2$ connected by a path;
- if $a \neq b$, then this path is simple;
- if $a = b$, then this path is a point or a simple loop;
- the trunk, $\cup_{\mathfrak{T}} \pi_{ab} \backslash \{a, b\}$, is a dense topological tree with degree at most 3.

[Lawler/Schramm/Werner 2004] established associated (unparametrised) Peano curve has SLE(8) scaling limit.

TIGHTNESS OF UST [j/w BARLOW/KUMAGAI]

Theorem. If $\mathbf{P}_\delta$ is the law of the measured, rooted spatial tree

$$\left(\mathcal{U}, \delta^{5/4} d_{\mathcal{U}}, \delta^2 \mu_{\mathcal{U}}(\cdot), \delta\phi_{\mathcal{U}}, 0\right)$$

under $\mathbf{P}$, then the collection $(\mathbf{P}_\delta)_{\delta\in(0,1)}$ is tight in $\mathcal{M}_1(\mathbb{T})$.

Proof involves:

- strengthening estimates of [Barlow/Masson],
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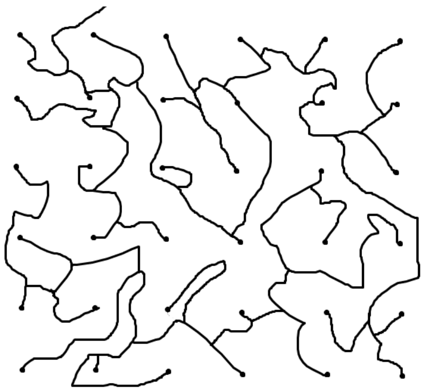

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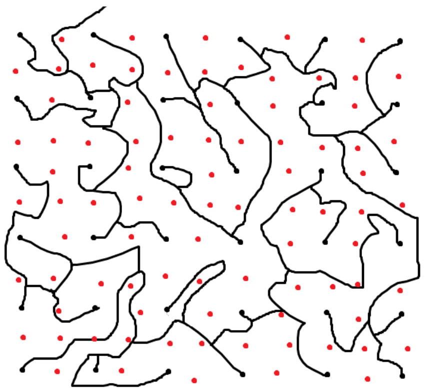

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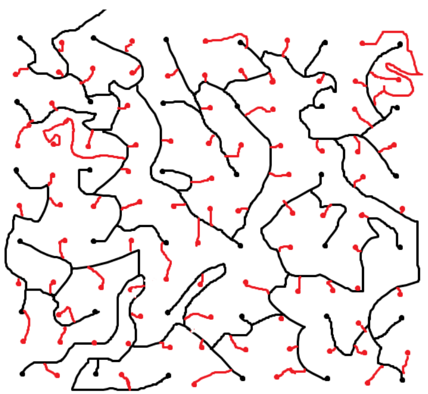


LIMITING PROCESS FOR SRW ON UST [j/w BARLOW/KUMAGAI]

Suppose $(\mathbf{P}_{\delta_i})_{i\geq 1}$, the laws of

$$\left(\mathcal{U}, \delta_i^{5/4} d_{\mathcal{U}}, \delta_i^2 \mu_{\mathcal{U}}, \delta_i \phi_{\mathcal{U}}, 0\right),$$

form a convergent sequence with limit $\tilde{\mathbf{P}}$.

Let $(\mathcal{T}, d_{\mathcal{T}}, \mu_{\mathcal{T}}, \phi_{\mathcal{T}}, \rho_{\mathcal{T}}) \sim \tilde{\mathbf{P}}$. It is then the case that $\mathbb{P}_{\delta_i}$, the annealed laws of

$$\left(\delta_i X^{\mathcal{U}}_{\delta_i^{-13/4} t}\right)_{t\geq 0},$$

converge to $\tilde{\mathbb{P}}$, the annealed law of

$$\left(\phi_{\mathcal{T}}(X^{\mathcal{T}}_t)\right)_{t\geq 0},$$

as probability measures on $C(\mathbb{R}_+, \mathbb{R}^2)$.

8. RANDOM WALK ON RANDOM PATHS

CRITICAL PERCOLATION BACKBONE

The backbone of the IIC is the union of all infinite simple paths from the origin.

The backbone pivotal bonds $(e_n)_{n\geq 1}$ are those edges that are used by all these paths.

Writing $e_n = (\underline{e}_n, \overline{e}_n)$, in high dimensions we have that

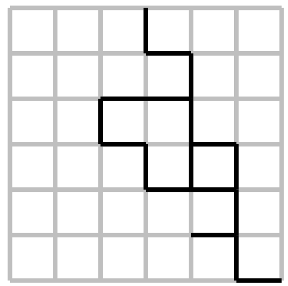
$$\left(n^{-1/2}\overline{e}_{nt}\right) \to (B_t)_{t\geq 0}.$$

Moreover, the backbone itself scales to range of Browian motion in the Hausdorff topology. [Heydenreich/van der Hofstad/Hulshof/Miermont]

(cf. Backbone of critical branching random walk conditioned to survive.)

RANGE OF A RANDOM WALK

Let $S = (S_n)_{n\in\mathbb{Z}}$ be the two-sided simple random walk on $\mathbb{Z}^d$ starting from 0, built on an underlying probability space with probability measure $\mathbf{P}$. Define the range of the random walk S to be the graph $\mathcal{G} = (V(\mathcal{G}), E(\mathcal{G}))$ with vertex set



$$V(\mathcal{G}) := \{S_n : n \in \mathbb{Z}\},$$

and edge set

$$E(\mathcal{G}) := \Big\{\{S_n, S_{n+1}\} : n \in \mathbb{Z}\Big\}.$$

For $\mathbf{P}$-a.e. random walk path, the graph $\mathcal{G}$ is infinite, connected and clearly has bounded degree.

DIFFUSIVE SCALING

Let $d \geq 5$. For $\mathbf{P}$-a.e. realisation of $\mathcal{G}$, the law of

$$\left(n^{-1/2}\mathrm{sgn}(X_{\lfloor tn \rfloor})(d_{\mathcal{G}}(0, X_{\lfloor tn \rfloor})\right)_{t \geq 0},$$

under $\mathbf{P}_0^{\mathcal{G}}$, converges as $n \to \infty$ to the law of $(B_{t\kappa_1(d)})_{t \geq 0}$. Furthermore, the law of

$$\left(n^{-1/4}X_{\lfloor tn \rfloor}\right)_{t \geq 0},$$

under $\mathbb{P}$, converges as $n \to \infty$ to the law of $(W^{(d)}_{B_{t\kappa_2(d)}})_{t \geq 0}$.

NB. Result does not hold in $d = 3, 4$ [C., Shiraishi].

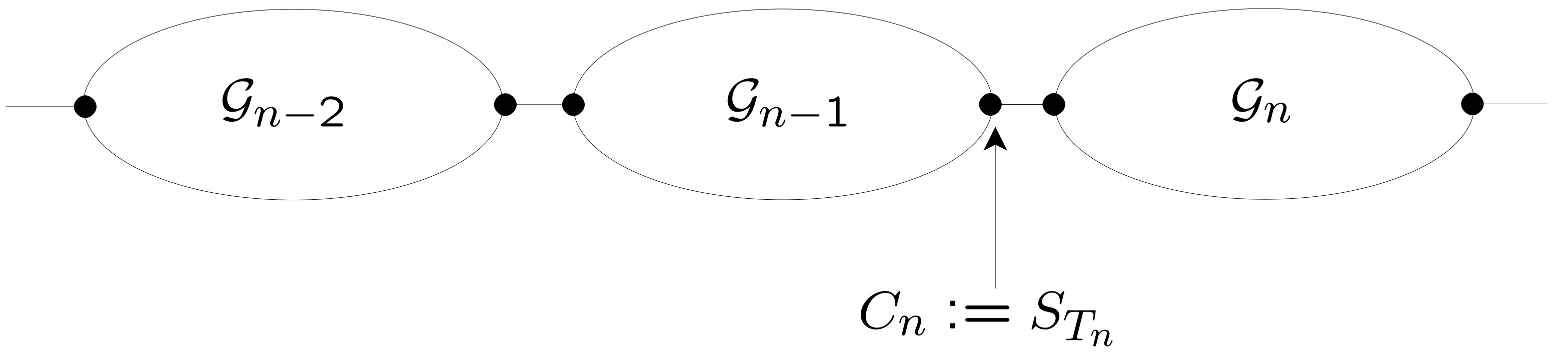
GEOMETRY OF TWO-SIDED RANGE

For $d \geq 5$, $\mathbf{P}$-a.s., the two-sided process S admits an infinite set of cut-times

$$\mathcal{T} := \{n : S_{(-\infty,n]} \cap S_{[n+1,\infty)} = \emptyset\},$$

which will be denoted $\dots T_{-2} < T_{-1} < T_0 \leq 0 < T_1 < T_2 < \dots$.

Under $\hat{\mathbf{P}} := \mathbf{P}(\cdot|0 \in \mathcal{T})$, $\mathcal{G}$ is made up of a string of stationary ergodic finite graphs.



[Bolthausen/Sznitman/Zeitouni] applied this kind of decomposition of the path of S to deduce the diffusivity of a random walk in a particular high-dimensional random environment.

JUMP PROCESS MARTINGALE

Define the hitting times of the set of cut-points $\mathcal{C} := \{C_n : n \in \mathbb{Z}\}$ by

$$H_0 := \inf\{m \geq 0 : X_m \in \mathcal{C}\},$$

and, for $n \geq 1$,

$$H_n := \inf\{m > H_{n-1} : X_m \in \mathcal{C}\}.$$

Let $J = (J_n)_{n \geq 0}$ be the $\mathbb{Z}$-valued process obtained by setting $J_n = m$ if $X_{H_n} = C_m$.

Elementary random walk/electrical network calculations imply that

$$\mathbf{P}_0^{\mathcal{G}}(J_{m+1} = n \pm 1 | J_m = n) = \frac{1}{\deg_{\mathcal{G}}(C_n) R_{\mathcal{G}}(C_n, C_{n \pm 1})}.$$

These formulae easily yield that the process

$$(\mathrm{sgn}(n) R_{\mathcal{G}}(0, C_{J_n}))_{n \geq 0}$$

is a martingale.

JUMP PROCESS CONVERGENCE

Applying the Lindeberg-Feller central limit theorem for martingales, we obtain

$$(n^{-1/2}\mathrm{sgn}(n)R_{\mathcal{G}}(0, C_{J_{\lfloor nt \rfloor}}))_{n\geq 0}$$

converges to a non-trivial Brownian motion.

Finally, ergodicity implies that $\hat{\mathbf{P}}$-a.s.

$$\mathrm{sgn}(n)R_{\mathcal{G}}(0, C_n) = \sum_{m=1}^{n} R_{\mathcal{G}}(C_{m-1}, C_m) \sim n\rho(d).$$

Similarly, $\hat{\mathbf{P}}$-a.s.,

$$\mathrm{sgn}(n)d_{\mathcal{G}}(0, C_n) = \sum_{m=1}^{n} d_{\mathcal{G}}(C_{m-1}, C_m) \sim n\delta(d).$$

Thus we can replace the resistance with the graph distance.

HITTING TIMES

The sequence

$$((\mathcal{G}-X_{H_n}, H_{n+1}-H_n))_{n\geq 0}$$

is ergodic under the annealed measure $\int \mathbf{P}_0^{\mathcal{G}}(\cdot)\mathrm{d}\hat{\mathbf{P}}$, where $\mathcal{G}-X_{H_n}$ is the graph with vertex set $\{x-X_{H_n} : x\in V(\mathcal{G})\}$ and edge set $\{\{x-X_{H_n}, y-X_{H_n}\} : \{x,y\}\in E(\mathcal{G})\}$.

It follows (after checking the necessary integrability) that, for $\hat{\mathbf{P}}$-a.e. realisation of $\mathcal{G}$, $\mathbf{P}_0^{\mathcal{G}}$-a.s.,

$$\frac{H_n}{n}\to \eta(d) := \frac{\hat{\mathbf{E}}\left(\mathrm{deg}_{\mathcal{G}}(0)\mathbf{E}_0^{\mathcal{G}}H_1\right)}{\hat{\mathbf{E}}\mathrm{deg}_{\mathcal{G}}(0)}\in[1,\infty).$$

TWO-SIDED SCALING LIMIT

Observe that $X_{H_n} \approx X_{\eta(d)n}$ and

$$\mathrm{sgn}(X_{H_n})d_{\mathcal{G}}(0, X_{H_n}) \approx \mathrm{sgn}(J_n)d_{\mathcal{G}}(0, C_{J_n}).$$

Diffusive scaling for $\mathrm{sgn}(X_n)d_{\mathcal{G}}(0, X_n)$ follows.

Similarly,

$$X_{H_n} \approx C_{J_n} \approx S_{T_{J_n}} \approx S_{\tau(d)J_n} \approx S_{\tau(d)\delta(d)^{-1}\mathrm{sgn}(J_n)d_{\mathcal{G}}(0,C_{J_n})},$$

which yields Euclidean scaling result.

(A time-change argument can be used to establish corresponding one-sided result.)

9. FURTHER PROPERTIES

HEAT KERNEL ESTIMATES FOR REAL TREES

Let $(\mathcal{T}, d_\mathcal{T}, \rho_\mathcal{T})$ be a compact, rooted real tree, and $\mu^\mathcal{T}$ a finite Borel measure on $\mathcal{T}$ with full support. Let $p_t^\mathcal{T}(x,y)$ be the heat kernel (transition density) of $X^\mathcal{T}$.

Suppose

$$\mu_\mathcal{T}(B_\mathcal{T}(x,r)) \asymp V(r)$$

for some function V satisfying the doubling condition $V(2r) \leq cV(r)$. Then, writing $h(r) = rV(r)$,

$$p_t^\mathcal{T}(x,y) \asymp \frac{c_1 h^{-1}(t)}{t} \exp\left\{ -\frac{d_\mathcal{T}(x,y)}{c_1 V^{-1}(t/d_\mathcal{T}(x,y))} \right\}.$$

The above is true for resistance forms in general when the resistance metric satisfies a chaining condition. [Kumagai 2004]

EXAMPLE: POLYNOMIAL VOLUME GROWTH

Suppose

$$\mu_{\mathcal{T}}(B_{\mathcal{T}}(x,r)) \asymp r^{\alpha}$$

Then,

$$p_t^{\mathcal{T}}(x,y) \asymp c_1 t^{-\frac{\alpha}{1+\alpha}} \exp\left\{ -\left(\frac{d_{\mathcal{T}}(x,y)^{\alpha+1}}{c_1 t} \right)^{\frac{1}{\alpha}} \right\}.$$

Note this is sub-Gaussian whenever $\alpha > 1$. In particular, the exit time of a ball of radius r about x satisfies:

$$E_x^{\mathcal{T}} \tau(x,r) \asymp r^{\alpha+1}.$$

VOLUME GROWTH FOR THE CRT

Recall the description of the Brownian CRT $\mathcal{T}$ in terms of the normalised Brownian excursion $(B_t)_{t\in[0,1]}$. We have

$$\mu_{\mathcal{T}}(\rho_{\mathcal{T}}, r) = \int_0^1 \mathbf{1}_{\{B_t<r\}} dt \geq \inf\{t : B_t = r\}.$$

Together with the invariance under rerooting of the CRT, and modulus of continuity results for Brownian excursion, it follows that almost-surely there exist (random) constants c_1, c_2, r_0 such that, for every $x \in \mathcal{T}$, $r \in (0, r_0)$,

$$c_1 r^2 \ell(1/r)^{-1} \leq \mu_{\mathcal{T}}(x, r) \leq c_2 r^2 \ell(1/r),$$

where $\ell(x) := \ln x \vee 1$. (Note that this implies the Hausdorff dimension of the CRT is 2.)

QUENCHED HEAT KERNEL ESTIMATES FOR THE CRT

The heat kernel of the Brownian CRT almost-surely satisfies, for some random constants c_1, c_2, c_3, c_4, $t_0 > 0$ and deterministic $\theta_1, \theta_2, \theta_3$,

$$p_t(x,y) \geq c_1 t^{-\frac{2}{3}} (\ell(t^{-1}))^{-\theta_1} \exp\left\{ -c_2 \left(\frac{d^3}{t}\right)^{1/2} \ell\left(\frac{d}{t}\right)^{\theta_2} \right\},$$

and

$$p_t(x,y) \leq c_3 t^{-\frac{2}{3}} (\ell(t^{-1}))^{1/3} \exp\left\{ -c_4 \left(\frac{d^3}{t}\right)^{1/2} \ell\left(\frac{d}{t}\right)^{-\theta_3} \right\},$$

for all $x, y \in \mathcal{T}$, $t \in (0, t_0)$, where $d := d_{\mathcal{T}}(x,y)$.

Moreover, fluctuations almost-surely actually occur.

UST LIMIT PROPERTIES

If $\tilde{\mathbf{P}}$ is a subsequential limit of $(\mathbf{P}_\delta)_{\delta\in(0,1)}$, then for $\tilde{\mathbf{P}}$-a.e. realisation of $(\mathcal{T}, d_\mathcal{T}, \mu_\mathcal{T}, \phi_\mathcal{T}, \rho_\mathcal{T})$ it holds that: given $R>0$, uniformly for $x\in B_\mathcal{T}(\rho_\mathcal{T},R)$ and $r\in(0,r_0)$,

$$c_1 r^{8/5}(\log r^{-1})^{-80}\leq \mu_\mathcal{T}\left(B_\mathcal{T}(x,r)\right)\leq c_2 r^{8/5}(\log r^{-1})^{80}$$

(so its Hausdorff dimension is 8/5). It follows that the heat kernel associated with the process $X^\mathcal{T}$ satisfies:

$$p_t^\mathcal{T}(x,y)\leq c_1 t^{-8/13}\ell(t^{-1})^{\theta_1}\exp\left\{-c_2\left(\frac{d_\mathcal{T}(x,y)^{13/5}}{t}\right)^{5/8}\ell(d_\mathcal{T}(x,y)/t)^{-\theta_2}\right\},$$

$$p_t^\mathcal{T}(x,y)\geq c_3 t^{-8/13}\ell(t^{-1})^{-\theta_3}\exp\left\{-c_4\left(\frac{d_\mathcal{T}(x,y)^{13/5}}{t}\right)^{5/8}\ell(d_\mathcal{T}(x,y)/t)^{\theta_4}\right\},$$

for all $x,y\in B_\mathcal{T}(\rho_\mathcal{T},R)$, $t\in(0,t_0)$

EXIT TIME BOUNDS

Under the assumptions of the previous slide:

$$E_x^{\mathcal{T}} \tau(x,r) \asymp h(r) = rV(r),$$

where $\tau(x,r) = \inf\{t > 0 : X_t^{\mathcal{T}} \in B_{\mathcal{T}}(x,r)^c\}$.

Proof: First, a Riesz representation argument allows us to write

$$E_x^{\mathcal{T}} \tau(x,r) = \int_{\mathcal{T}} g_B(x,y)\mu_{\mathcal{T}}(dy),$$

where the Green kernel g_B satisfies

$$\mathcal{E}_{\mathcal{T}}(g_B(x,\cdot),f) = f(x),$$

for all $f \in \mathcal{F}_{\mathcal{T}}$ with $f|_{B^c} = 0$. Standard arguments then imply that $g_B(x,x) > 0$ and $p(y) := g_B(x,y)/g_B(x,x)$ is an equilibrium kernel, i.e.

$$\mathcal{E}_{\mathcal{T}}(p,p) = \inf\{\mathcal{E}_{\mathcal{T}}(f,f) : f \in \mathcal{F}, f(x) = 1, f|_{B^c} = 0\} = R_{\mathcal{T}}(x, B_{\mathcal{T}}(x,r)^c)^{-1}.$$

PROOF OF EXIT TIME BOUNDS (CONT.)

From the above properties, we deduce that

$$g_B(x,x) = R_{\mathcal{T}}(x, B_{\mathcal{T}}(x,r)^c) \asymp r.$$

Hence

$$E_x^{\mathcal{T}} \tau(x,r) = \int_{\mathcal{T}} g_B(x,y)\mu_{\mathcal{T}}(dy) \leq g_B(x,x)\mu_{\mathcal{T}}(B_{\mathcal{T}}(x,r)) \asymp rV(r).$$

For the lower bound: if $y \in B_{\mathcal{T}}(x, \varepsilon r)$, then

$$|p(x) - p(y)|^2 \leq \mathcal{E}_{\mathcal{T}}(p,p) d_{\mathcal{T}}(x,y) = R_{\mathcal{T}}(x, B_{\mathcal{T}}(x,r)^c)^{-1} \varepsilon r \leq \frac{1}{4}$$

for ε suitably small. Hence

$$E_x^{\mathcal{T}} \tau(x,r) \geq \int_{B_{\mathcal{T}}(x,\varepsilon r)} g_B(x,y)\mu_{\mathcal{T}}(dy) \geq \frac{1}{2} g_B(x,x)\mu_{\mathcal{T}}(B_{\mathcal{T}}(x,\varepsilon r)) \asymp rV(r).$$