Multifractal formalisms for the local spectral and walk dimensions

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Abstract

We introduce the concepts of local spectral and walk dimension for fractals. For a class of finitely ramified fractals we show that, if the Laplace operator on the fractal is defined with respect to a multifractal measure, then both the local spectral and walk dimensions will have associated non-trivial multifractal spectra. The multifractal spectra for both dimensions can be calculated and are shown to be transformations of the original underlying multifractal spectrum for the measure, but with respect to the effective resistance metric.

1 Introduction

Multifractal analysis was introduced in the physics literature (see for example [10]) to provide a finer description of fractal phenomena which displayed a range of power law scalings. A mathematically rigorous version has been developed in a number of papers, see for example [5], [6], [1], [17], [18]. We give a brief

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discussion. The local dimension of a measure μ at a point x in some fractal set K is defined to be, if the limit exists,

$$\dim_{loc}(x) = \lim_{r \to 0} \frac{\log \mu(B_r(x))}{\log r},$$

where $B_r(x)$ is a ball of radius r about x. The sets of interest in multifractal analysis are $E_\alpha = \{x \in K : \dim_{loc}(x) \text{ exists and equals } \alpha\}$. The multifractal spectrum is then defined to be the function $f(\alpha) = \dim(E_\alpha)$, for some notion of dimension. A multifractal formalism has been developed which relates the function $f(\alpha)$ to the Legendre transform of the moment sums of the measure.

Analysis on fractals has shown that there are other 'dimensions' which have an essential role to play in the study of the Laplace operator on a fractal. We will consider firstly the spectral dimension d_s (or fracton dimension), which characterises the scaling in the eigenvalue counting function, $N(x)$, for the Laplace operator on the fractal, in that

$$d_s = \lim_{x \to \infty} \frac{2 \log N(x)}{\log x}.$$

This definition shows that the spectral dimension is a global property of the fractal set (for a discussion of the behaviour of this counting function see [16]). Secondly we tackle the walk dimension d_w, the scaling exponent for the mean square displacement of a diffusing particle (see [13] for a physical discussion).

We will define local versions of these dimensions for a Laplace operator with respect to a self-similar measure and develop a corresponding multifractal analysis for a class of finitely ramified fractals. In order to give the spectral dimension a local interpretation we use the short time asymptotics of the heat kernel at a particular point in the fractal. Let P_t be the transition semigroup generated by the Laplace operator on the fractal. This has a density (the heat kernel) with respect to the self-similar measure, denoted by $p_t(x, y)$. If we consider the diagonal of this density, then, in the case of fractals such as the Sierpinski gasket, when the self-similar measure is the Hausdorff measure, it is known that there exist constants $0 < c_1, c_2$ such that,

$$c_1 t^{-d_s/2} \leq p_t(x, x) \leq c_2 t^{-d_s/2} \quad \text{for all} \quad 0 < t < 1, \;\; x \in K.$$

If the set does not have as many symmetries as the Sierpinski gasket, or the Laplace operator is defined with respect to a measure which has a non-trivial multifractal spectrum, the exponent in these upper and lower bounds may depend on the particular point x. This leads to the idea of the *local spectral dimension*, defined (when it exists) as

$$d_s(x) = \lim_{t \downarrow 0} \frac{2 \log p_t(x,x)}{-\log t}. \tag{1.1}$$

Note that as the Laplace-Stieltjes transform of the eigenvalue counting measure is the trace of the heat semigroup there is a relationship between the global spectral dimension and the local spectral dimension.

To define the local walk dimension, we will consider $T_{B_r(x)}$, the time taken for the diffusion, whose infinitesimal generator is the Laplace operator (with respect to the self-similar measure) on the fractal, to exit $B_r(x)$. Then the *local walk dimension* is defined to be (when it exists),

$$d_w(x) = \lim_{r \to 0} \frac{\log E^x(T_{B_r(x)})}{\log r}. \tag{1.2}$$

As in the case of the usual multifractal analysis we can ask about the dimension of the subsets of the fractal for which there is a particular local spectral or walk dimension. Let

$$J_\gamma = \{x \in K : d_s(x) \text{ exists and equals } \gamma\}, \tag{1.3}$$

and define the multifractal spectrum for the local spectral dimension to be $g(\gamma) = \dim_H(J_\gamma)$, where we denote the Hausdorff dimension by $\dim_H$. For the local walk dimension we can consider sets

$$W_\gamma = \{x \in K : d_w(x) \text{ exists and equals } \gamma\}, \tag{1.4}$$

and define the corresponding multifractal spectrum to be $h(\gamma) = \dim_H(W_\gamma)$.

In this paper we will consider a class of finitely ramified fractals, called p.c.f. self-similar sets and show that both of these multifractal spectra are well defined. A p.c.f. self-similar set K consists of a triple $(K, S, \{F_s\}_{s \in S})$ where each F_s is a contraction map on K such that $K = \cup_{i \in S} F_i(K)$ and $S = \{1, 2, \cdots, N\}$ is a set of indices for the contraction maps. Typical examples are as follows.

Example 1.1 *(1) Line segment:*

$$K = [0,1];\ S = \{1,2\};\ F_1(x) = x/2,\ F_2(x) = x/2 + 1/2 \quad (x \in \mathbf{R}).$$

(2) Sierpinski gasket:

$$K = 2\text{-}dimensional\ Sierpinski\ gasket\ (Figure\ 1);\ S = \{1, 2, 3\};$$
$$F_1(x) = x/2,\ F_2(x) = (x - a)/2 + a,\ F_3(x) = (x - b)/2 + b \quad (x \in \mathbf{R}^2),$$

where we set $a = (1, 0),\ b = (1/2, \sqrt{3}/2)$. *Similarly, the* n*-dimensional Sierpinski gasket is a p.c.f. self-similar set.*

(3) Tree-like set:

$$K = Figure\ 2;\ S = \{1, 2\};\ F_1(z) = \beta\bar{z},\ F_2(z) = (1 - |\beta|^2)\bar{z} + |\beta|^2 \quad (z \in \mathbf{C}),$$

where we set β *such that* $|\beta| < 1,\ |1 - \beta| < 1$ *and Im* $\beta \neq 0$.

Figure 1 here

Figure 1: Sierpinski gasket

Figure 2 here

Figure 2: Tree-like set

We will show that the above multifractal spectra can be obtained as simple transformations of the multifractal spectrum (in the resistance metric) of the underlying measure.

Let μ be a Bernoulli measure on the fractal with scale factors $\{\mu_i : \sum_i \mu_i = 1\}$ which has a multifractal spectrum $f(\alpha)$. Our main result, Theorem 3.4, shows that if the Laplace operator with respect to μ is constructed from a regular harmonic structure, then the multifractal spectrum of the local spectral dimension is given by

$$g(\gamma) = f(\frac{\gamma}{2 - \gamma}) \text{ for } \frac{\gamma}{2 - \gamma} \in \text{Supp}(f). \tag{1.5}$$

We note that for p.c.f. fractals with regular harmonic structure the local spectral dimension is always less than 2. We also show in Theorem 4.1 that under the same assumptions the multifractal spectrum for the local walk dimension is given by

$$h(\gamma) = f(\gamma - 1) \ \text{ for } \ \gamma - 1 \in \text{Supp}(f). \tag{1.6}$$

The multifractal analysis of the exponents for more general fractals and non-regular p.c.f. fractals will be treated in [11]. The techniques that we use here are a simple extension of those used in the usual multifractal analysis of measures coupled with precise local estimates on the heat kernel.

We can relate the multifractal spectra g, h to the scaling in the moment measures. If the harmonic structure is regular with resistance scale factors $\rho_i > 1$ (for all $i \in S$), we can define $\beta(q)$ as the unique real number such that

$$\sum_i \mu_i^q \rho_i^{-\beta(q)} = 1. \tag{1.7}$$

The multifractal spectrum of the measure (in the resistance metric) is given by a Legendre transform $f(\alpha) = \inf_q\{\beta(q) + \alpha q\}$. The multifractal spectra for the spectral and walk dimensions can also be obtained by taking a suitable Legendre transform of β and these results are given in Theorems 3.4 and 4.1.

Note that by the Einstein relation (see [13]) $d_w = d_f + d_r$, where d_r is a resistance exponent, hence in the resistance metric we anticipate that $d_w = d_f + 1$. However, at the local level there exist points x such that $d_w(x) \neq d_f(x) + 1$. The relationship between the spectral, walk and fractal dimensions is that $d_s/2 = d_f/d_w$ and this shows heuristically why the transformations are of the above form.

For the case of the line segment (Example 1.1 1)), our main result can be explained as follows. For $0 < p < 1$, let μ be a Bernoulli measure on $[0,1]$ such that $\mu(F_{i_1} \circ \cdots F_{i_n}([0,1])) = \mu_{i_1} \cdots \mu_{i_n}$ for all $i_1, \cdots i_n \in \{1,2\}$ where $\mu_1 = p,\ \mu_2 = 1-p$ (this measure is called the de Rham p-measure). When $p = 1/2$, it is the Lebesgue measure on $[0,1]$. Let X_t be a diffusion process which corresponds to the self-adjoint operator $\frac{d}{d\mu}\frac{d}{dx}$ on $\mathbf{L}^2([0,1],\mu)$. This process can be considered as a (singular) time changed Brownian motion on $[0,1]$. Define the local spectral dimension and the local walk dimension for this process as in (1.1), (1.2), and set J_γ, W_γ as in (1.3), (1.4). Using the moment scalings we can set

$$\beta(q) = \frac{\log(p^q + (1-p)^q)}{\log 2},$$

(from (1.7) with $\rho_1 = \rho_2 = 2$) and find an explicit formula for the multifractal spectrum of the de Rahm p-measure,

$$f(\alpha) = \frac{\log((\frac{-\log((1-p)2^\alpha)}{\log(p2^\alpha)})^\eta + (\frac{-\log((1-p)2^\alpha)}{\log(p2^\alpha)})^{\eta-1})}{\log 2} + \alpha\frac{\log(\frac{-\log((1-p)2^\alpha)}{\log(p2^\alpha)})}{\log p/(1-p)},$$

where $\eta = \log(p)/\log(p/(1-p))$. Using the transformation of f in (1.5), (1.6) we have explicit formulas for g, h and

$$\dim_H J_\gamma = g(\gamma) \vee 0, \qquad \dim_H W_\gamma = h(\gamma) \vee 0.$$

Here $\dim_H$ is the Hausdorff dimension with respect to the resistance metric (which is equivalent to the Euclidean metric in this case). When $p = 1/2$, $g(\gamma) \vee 0 = \delta_1(\gamma)$ and $h(\gamma) \vee 0 = \delta_2(\gamma)$, which agrees with the fact that $d_s \equiv 1, d_w \equiv 2$ for Brownian motion. The main point of our result is that when $p \neq 1/2$, the local behavior of $d_s(x)$ and $d_w(x)$ are highly dependent on x and we can describe some of this dependence through the multifractal analysis presented here.

The paper is organized as follows. In Section 2 we introduce the class of fractals that we will consider and define their Laplace operators and diffusion processes via Dirichlet forms. We also state some transition density estimates which we discuss in more detail in the appendices. The result for the local spectral dimension is given in Section 3 and for the local walk dimension in Section 4.

2 Dirichlet forms on p.c.f. self-similar sets and their heat kernel estimates

In this section, we briefly introduce p.c.f. self-similar sets, an abstract formulation of finitely ramified fractals. We also introduce their Dirichlet forms, as there is a one to one correspondence (under some conditions) between the Dirichlet forms and the diffusion processes and Laplace operators on these sets. Finally we will state on-diagonal transition density estimates for the associated diffusion process. For more details and discussions of these sets we refer the reader to [14], [2]. Readers who are not familiar with this kind of abstract setting may just consider the concrete examples indicated in Example 1.1.

We first introduce the notion of a self-similar structure.

Definition 2.1 *Let K be a compact metrizable space and for each $s \in S \equiv \{1, 2, \cdots, N\}$, let $F_s : K \to K$ be a continuous injection. The triple $\mathcal{L} = (K, S, \{F_s\}_{s \in S})$ is said to be a self-similar structure on K if there exists a continuous surjection $\pi : S^{\mathbf{N}} \to K$ such that $\pi \circ \tilde{\sigma}_s = F_s \circ \pi$ for every $s \in S$. Here $\tilde{\sigma}_s : S^{\mathbf{N}} \to S^{\mathbf{N}}$ is defined as $\tilde{\sigma}_s w = sw$.*

In this paper, for each $w \in S^{\mathbf{N}}$ we denote the i-th element in the sequence by w_i and write $w = w_1 w_2 w_3 \cdots$. For $w \in S^n$, we write $F_w = F_{w_1} \circ F_{w_2} \circ \cdots \circ F_{w_n}$ and $K_w = F_w(K)$. In particular, $K_s = F_s(K)$ for $s \in S$. Note that an iterated function system and its invariant set (see [6], p 29) is a self-similar structure in the sense of Definition 2.1 by taking $\pi(w) = \cap_{n \geq 1} F_{w_1} \circ F_{w_2} \circ \cdots \circ F_{w_n}(K)$.

Definition 2.2 *Let $\mathcal{L} = (K, S, \{F_s\}_{s \in S})$ be a self-similar structure on K. The critical set of $\mathcal{L}$ is defined by $C(\mathcal{L}) = \pi^{-1}(\cup_{s,t \in S, s \neq t}(K_s \cap K_t))$ and the post critical set of $\mathcal{L}$ is defined by $P(\mathcal{L}) = \cup_{n \geq 1} \sigma^n(C(\mathcal{L}))$.*
$\mathcal{L}$ is called post critically finite (p.c.f. for short) if $P = P(\mathcal{L})$ is a finite set.

Thus P is the set of addresses of all pre-images of intersection points and, as this is finite, we see that p.c.f. self-similar sets correspond roughly to the class of finitely ramified fractals as used in the physics literature. Throughout the paper we will only consider the p.c.f. self-similar sets which are connected. For $m \geq 0$, let

$$P^{(m)} = \cup_{w \in S^m} wP, \quad V_m = \pi(P^{(m)}) \quad \text{and} \quad V_* = \cup_{m \geq 0} V_m.$$

Thus V_m is a set of points approximating the fractal and V_0 is the boundary of the fractal. Let μ be a self-similar measure on K with $\mu(F_i(K)) = \mu_i > 0$ for $i \in S$ where μ_i satisfies $\sum_i \mu_i = 1$.

In [14], one of the authors introduced the notion of p.c.f. self-similar sets and constructed local regular Dirichlet forms $(\mathcal{E}, \mathcal{F})$ on them. In order to construct a Dirichlet form, the existence of a regular harmonic structure was assumed. That is the existence of a non-degenerate fixed point for a non-linear renormalization map. There are many fractals which satisfy this assumption (typical examples are those in Example 1.1). We will make this assumption for p.c.f. self-similar sets and introduce the results we need concerning the basic properties of their Dirichlet forms.

The resistance $R(p,q)$ between points $p,q \in K$ can be defined using the Dirichlet form $(\mathcal{E},\mathcal{F})$, by

$$R(p,q) = \left(\inf\{\mathcal{E}(f,f) : f(p)=0, f(q)=1\}\right)^{-1},$$

where we set $\inf \emptyset = \infty$. The function $R(\cdot,\cdot)$ determines a metric, which we call the (effective) resistance metric, on K.

Theorem 2.3 ([14],[15]) *There exists a local regular Dirichlet form $(\mathcal{E},\mathcal{F})$ on $\mathbf{L}^2(K,\mu)$ which has the following property:*

$$|f(p)-f(q)|^2 \le R(p,q)\mathcal{E}(f,f) \qquad \textit{for} \quad f\in\mathcal{F}, \quad p,q\in K, \tag{2.1}$$

$$\mathcal{E}(f,g) = \sum_{i=1}^{N} \rho_i \mathcal{E}(f\circ F_i, g\circ F_i) \qquad \textit{for} \quad f,g\in\mathcal{F}, \tag{2.2}$$

where $\rho_i > 1$ $(i\in S)$. Further, if we set $\mathcal{E}_{(\beta)}(\cdot,\cdot) = \mathcal{E}(\cdot,\cdot)+\beta(\cdot,\cdot)_{\mathbf{L}^2(K,\mu)}$ for $\beta>0$, then, $\mathcal{E}_{(\beta)}$ admits a positive symmetric continuous reproducing kernel $g^K_\beta(\cdot,\cdot)$.

It is not difficult to show the following approximate scaling property

$$c_{2.1}\rho_w^{-1}R(p,q) \le R(F_w(p),F_w(q)) \le c_{2.2}\rho_w^{-1}R(p,q), \tag{2.3}$$

for all $p,q\in K,\ w\in S^m,\ m\ge 0$ for some constants $c_{2.1},c_{2.2}>0$. Here and in the following, we will write $\rho_w = \rho_{w_1}\rho_{w_2}\cdots\rho_{w_m}$ for $w=w_1w_2\cdots w_m$.

For the cases of Example 1.1, one can construct Dirichlet forms satisfying (2.1) and (2.2) with the following ρ_i's. 1) line segment: $\rho_1=\rho_2=2$. (The classical Dirichlet integral $\mathcal{E}(f,g)=\frac{1}{2}\int_0^1 \nabla f(x)\nabla g(x)dx$ together with the classical Sobolev space with the Neumann boundary condition is the corresponding Dirichlet form.) 2) Sierpinski gasket: $\rho_i = 5/3,\ i\in\{1,2,3\}$ (see [14]). 3) Tree-like set: $\rho_1 = 1/\alpha,\ \rho_2 = 1/(1-\alpha^2)$ for any $0<\alpha<1$ (see [14]).

We now give a rough outline of the construction of the Dirichlet form. The first step is to construct a sequence of Dirichlet forms $\mathcal{E}_m$ on $l(V_m)\equiv\{f : V_m \to \mathbf{R}\}$ which has a consistency property and is monotone increasing w.r.t. m (this is a form of decimation invariance and follows from our fixed point assumption).

The limit $(\mathcal{E}, \mathcal{F})$ is then defined firstly as a form on $l(V_*)$ by

$$\begin{aligned} \mathcal{F} &= \{f \in l(V_*) : \lim_{m\to\infty} \mathcal{E}_m(f|_{V_m}, f|_{V_m}) < \infty\}, \\ \mathcal{E}(f,g) &= \lim_{m\to\infty} \mathcal{E}_m(f|_{V_m}, g|_{V_m}) \quad \text{for} \quad f, g \in \mathcal{F}, \end{aligned}$$

and then, using the metric R, $\mathcal{F}$ is embedded into the space of continuous functions on K. It can be shown that this is a subspace of $\mathbf{L}^2(K,\mu)$ for any Borel measure μ. The Laplace operator can then be obtained from the Gauss-Green formula. For more details see [14] and [15]. We remark again that there is a one to one correspondence between local regular Dirichlet forms and diffusion processes (up to quasi-everywhere starting point), [9]. The corresponding diffusion process $\{X^x(t)\}$ ($x \in K$ is the starting point) has the following self-similarity property,

$$X^{F_i(x)}(t/t_i) \stackrel{d}{=} F_i(X^x(t)) \quad \text{for} \;\; x \in K,\; i \in S,\; 0 \le t \le \inf\{s \ge 0 : X^x(s) \in V_0\},$$

where $t_i = \rho_i/\mu_i$ and $X \stackrel{d}{=} Y$ means X and Y have the same law.

In [4], one of the authors obtained estimates on the short time behaviour of the transition density of the diffusion corresponding to $(\mathcal{E}, \mathcal{F})$ on $\mathbf{L}^2(K,\mu)$. (Note that the transition density is the fundamental solution of the heat equation). Here we state part of this result; more details can be found in Appendix A and B. Let $\hat{\mu}$ be any Bernoulli measure on K so that $\hat{\mu}(F_i(K)) = \hat{\mu}_i$ $(i \in S)$ and $\hat{\mu}(V_0) = 0$ where $\hat{\mu}_i \ge 0$ satisfies $\sum_i \hat{\mu}_i = 1$. Also define $d_s(\hat{\mu})/2$ as follows,

$$d_s(\hat{\mu})/2 = \frac{\sum_i \hat{\mu}_i \log(1/\mu_i)}{\sum_i \hat{\mu}_i \log(\rho_i/\mu_i)}. \tag{2.4}$$

Theorem 2.4 ([4]) *There exists a jointly continuous heat kernel $p_t(x,y)$ of $(\mathcal{E}, \mathcal{F})$ on $\mathbf{L}^2(K,\mu)$ such that*

$$\lim_{t\to 0} \frac{\log p_t(x,x)}{\log t} = -d_s(\hat{\mu})/2 \qquad \textit{for} \quad \hat{\mu}\textit{-a.e. } x \in K.$$

3 Multifractal structure of the local spectral dimension

3.1 The multifractal analysis of self-similar measures

In this subsection, we recall the usual multifractal analysis of self-similar measures in the context of p.c.f. self-similar sets with regular harmonic structures. See Section 11 of [6] for a description of the idea of multifractal analysis. In [6] (also in Theorem 3.2 of [5]), a strong separation condition is assumed on K and the objects treated are Cantor like sets. However, as in [1], we can work with the case when the components just touch, by relaxing the strong separation condition to the so called open set condition, that our p.c.f. self-similar sets satisfy (the condition in [1] corresponds to the open set condition for deterministic self-similar sets). Let $B(x,\delta) = \{p \in K : R(p,x) < \delta\}$ denote the ball of radius δ in the resistance metric and let $\mathrm{diam}_R(A)$ denote the diameter of the set A with respect to R. Recall that for each $w \in S^n$ we have

$$c_1(\rho_w)^{-1} \le \mathrm{diam}_R(K_w) \le c_2(\rho_w)^{-1} \quad \text{and} \quad \mu(K_w) = \mu_w,$$

where we denote $\rho_w = \rho_{w_1 \cdots w_n}$ and $\mu_w = \mu_{w_1 \cdots w_n}$. For each $q \in \mathbf{R}$, define $\beta = \beta(q)$ as the (unique) real number satisfying

$$\sum_{i=1}^{N} \mu_i^q \rho_i^{-\beta(q)} = 1. \tag{3.1}$$

Now, define $f : [\alpha_{\min}, \alpha_{\max}] \to \mathbf{R}$ to be the Legendre transform of β, given by

$$f(\alpha) = \inf_{-\infty<q<\infty} \{\beta(q) + \alpha q\}, \tag{3.2}$$

where $-\alpha_{\min}, -\alpha_{\max}$ are the slopes of the asymptotes of the convex function β. We also assume that $\log(1/\mu_i)/\log \rho_i$ is not the same for all $i \in S$, as otherwise the following result (Theorem 3.3) is easy to deduce. In this case β is strictly convex and the infimum in (3.2) is attained at a unique $q = q(\alpha)$. By differentiation, this occurs when

$$\alpha = -\frac{d\beta}{dq}, \tag{3.3}$$

so that $f(\alpha(q)) = \beta(q) + \alpha q$. Note that if any one of the three quantities $q \in \mathbf{R}, \beta \in \mathbf{R}$ and $\alpha \in (\alpha_{\min}, \alpha_{\max})$ is given, the other two are determined by (3.1) and (3.3). In particular,

$$\alpha = \frac{\sum_i \mu_i^q \rho_i^{-\beta(q)} \log(1/\mu_i)}{\sum_i \mu_i^q \rho_i^{-\beta(q)} \log \rho_i}. \tag{3.4}$$

We thus see that

$$\alpha_{\min} = \min_i \frac{\log(1/\mu_i)}{\log \rho_i} \quad \text{and} \quad \alpha_{\max} = \max_i \frac{\log(1/\mu_i)}{\log \rho_i}. \tag{3.5}$$

For given q and $\beta = \beta(q)$, define a probability measure ν on K by

$$\nu(K_w) = \mu_w^q \rho_w^{-\beta}, \tag{3.6}$$

for each $w \in S^n,\ n \in \mathbf{N}$. For each $\alpha \geq 0$, define

$$\begin{aligned} E_\alpha &= \{x \in K : \lim_{r\to 0} \log(\mu(B(x,r))/\log r = \alpha\}, \\ \hat{E}_\alpha &= \pi(\{w \in S^{\mathbf{N}} : \lim_{n\to\infty} \frac{\sum_{i=1}^n \log(1/\mu_{w_i})}{\sum_{i=1}^n \log \rho_{w_i}} = \alpha\}), \end{aligned}$$

where $\pi : S^{\mathbf{N}} \to K$ is the surjection defined in Definition 2.1. By the strong law of large numbers and (3.4), we have $\nu(\hat{E}_\alpha) = 1$. Also, for each $x = \pi(w) \in \hat{E}_\alpha$,

$$\lim_{n\to\infty} \frac{\log(\nu(K_{w_1\cdots w_n}))^{-1}}{\log \rho_{w_1\cdots w_n}} = \beta(q) + \alpha q = f(\alpha(q)). \tag{3.7}$$

For $w \in S^{\mathbf{N}}$, denote $w|n = w_1 \cdots w_n$. Let

$$A_\nu = \{x = \pi(w) \in K : \lim_{r\to 0} \frac{\log(\nu(B(x,r))}{\log r} = \lim_{n\to\infty} \frac{\log(\nu(K_{w|n}))^{-1}}{\log \rho_{w|n}}\},$$

and define A_μ in the same way by changing ν to μ.

We prepare two basic lemmas for later use. See [1], [5], [6], [11] for the proof.

Lemma 3.1 $\nu(A_\nu) = \nu(A_\mu) = 1$

Lemma 3.2 *Let $E \subset K$ and let μ be a finite measure. If $\mu(E) > 0$ and there is a constant $a > 0$ such that*

$$\lim_{r\to 0} \log \mu(B(x,r))/\log r = a, \qquad \textit{for all } x \in E,$$

then $\dim_H E = a$ and μ is the Hausdorff measure on E (up to a constant multiple).

We are now ready to explain a theorem on the multifractal analysis of self-similar measures on p.c.f self-similar sets. In the case when $\log(1/\mu_i)/\log\rho_i$ is the same for all $i \in S$ (therefore $\alpha_{\min} = \alpha_{\max} \equiv \alpha_0$), we will define $f(\alpha) = (\dim_H K)\delta_{\alpha_0}(\alpha)$. Then the following holds.

Theorem 3.3 *If* $\alpha \in [\alpha_{\min}, \alpha_{\max}]$*, then* $\dim_H E_\alpha = f(\alpha)$*. If* $\alpha \notin [\alpha_{\min}, \alpha_{\max}]$*, then* $\dim_H E_\alpha = 0$*.*

PROOF: For $\alpha \in (\alpha_{\min}, \alpha_{\max})$, the proof of $\dim_H E_\alpha \geq \dim_H(E_\alpha \cap \hat{E}_\alpha \cap A_\nu) = f(\alpha)$ is straightforward using (3.7), Lemma 3.1, Lemma 3.2 and the fact $\nu(\hat{E}_\alpha) = 1$. $\dim_H E_\alpha \leq f(\alpha)$ can be proved in the same way as Lemma 4.2 of [1]. The case $\alpha \notin (\alpha_{\min}, \alpha_{\max})$ is also routine (see, for example, Lemma 4.3 of [1] for $\alpha \notin [\alpha_{\min}, \alpha_{\max}]$ and Theorem 2.14 of [5] for $\alpha \in \{\alpha_{\min}, \alpha_{\max}\}$). ∎

3.2 The main theorem

In this subsection, we will prove our main theorem on the multifractal structure of the local spectral dimension for diffusion processes with respect to regular harmonic structures on p.c.f. self-similar sets. In order to do this we must consider the set of points with the same local spectral dimension, thus for each $\gamma \geq 0$, let

$$J_\gamma = \{x \in K : -\lim_{t\to 0} \frac{2\log p_t(x,x)}{\log t} = \gamma\}. \tag{3.8}$$

Also we require a suitable spectrum and for this purpose we extend the multifractal spectrum f for the measure defined in (3.2) to a function $\bar{f}$ on the real line by setting

$$\bar{f}(\alpha) = \begin{cases} f(\alpha), & \alpha \in [\alpha_{\min}, \alpha_{\max}], \\ 0, & \text{otherwise.} \end{cases} \tag{3.9}$$

Now define our candidate for the spectrum by $g(\gamma) = \bar{f}(\gamma/(2-\gamma))$. We note that

$$\text{Supp } g = [\frac{2\alpha_{\min}}{\alpha_{\min}+1}, \frac{2\alpha_{\max}}{\alpha_{\max}+1}] \subset (0,2),$$

where the last inclusion is due to the regularity of the harmonic structure we assumed in the construction of the Dirichlet form.

We are now ready to state the main result of this paper which provides a multifractal formalism for the local spectral dimension.

Theorem 3.4 *1. Let K be a p.c.f. self-similar fractal with a regular harmonic structure, then*

$$\dim_H J_\gamma = g(\gamma) \quad \textit{for} \quad \gamma \geq 0.$$

2. Let $\beta(q)$ be defined as the number such that

$$\sum_i \mu_i^q \rho_i^{-\beta(q)} = 1.$$

The multifractal spectrum is given by

$$g(\gamma) = \inf\{\beta(q) + \frac{\gamma}{2-\gamma} q\} \quad \textit{for} \quad \gamma \in [\frac{2\alpha_{\min}}{\alpha_{\min}+1}, \frac{2\alpha_{\max}}{\alpha_{\max}+1}].$$

As we will see, the proof follows from the standard techniques of multifractal analysis. We first show the lower estimate. Set $\gamma(\alpha) = 2\alpha/(1+\alpha)$ for each $\alpha \geq 0$ so that $g(\gamma(\alpha)) = \bar{f}(\alpha)$.

Proposition 3.5

$$\dim_H J_{\gamma(\alpha)} \geq f(\alpha) \quad \textit{for} \quad \alpha \in (\alpha_{\min}, \alpha_{\max}).$$

Proof. We consider the measure ν constructed from the set of weights $\{\mu_i\}$ as in (3.6). Recall that for each given $q \in \mathbf{R}$ we have (3.4). On the other hand, if we take $\hat{\mu} = \nu$ in Theorem 2.4, then, by (2.4), we have

$$d_s(\nu) = \frac{2\sum_i \mu_i^q \rho_i^{-\beta} \log(1/\mu_i)}{\sum_i \mu_i^q \rho_i^{-\beta} \log(\rho_i/\mu_i)} = 2\alpha/(1+\alpha).$$

Thus, by Theorem 2.4, we have

$$\nu(J_{\gamma(\alpha)}) = 1.$$

As $J_{\gamma(\alpha)} \cap \hat{E}_\alpha \cap A_\nu \subset J_{\gamma(\alpha)}$, we obtain $\dim_H J_{\gamma(\alpha)} \geq \dim_H(J_{\gamma(\alpha)} \cap \hat{E}_\alpha \cap A_\nu) = f(\alpha)$ using Lemma 3.1 and Lemma 3.2. ∎

We next show the upper estimate. As mentioned in the last subsection, $\alpha \in (\alpha_{\min}, \alpha_{\max})$ and γ are determined by $q \in \mathbf{R}$. We thus consider $\alpha = \alpha(q), \gamma = \gamma(q)$ as functions of q. For $q < 0$, set

$$\begin{aligned} \bar{L}_q &= \{x \in K \setminus V_* : \liminf_{t \to 0} -\frac{2 \log p_t(x,x)}{\log t} \geq \gamma(q)\}, \\ L_q &= \pi(\{w \in S^{\mathbf{N}} : \liminf_{n \to \infty} \frac{\sum_{i=1}^n \log(1/\mu_{w_i})}{\sum_{i=1}^n \log \rho_{w_i}} \geq \alpha(q)\}). \end{aligned}$$

For $q \geq 0$, set

$$\bar{U}_q = \{x \in K \setminus V_* : \limsup_{t\to 0} -\frac{2\log p_t(x,x)}{\log t} \leq \gamma(q)\},$$
$$U_q = \pi(\{w \in S^{\mathbf{N}} : \liminf_{n\to\infty} \frac{\sum_{i=1}^n \log(1/\mu_{w_i})}{\sum_{i=1}^n \log \rho_{w_i}} \leq \alpha(q)\}).$$

Note that the definition of $\bar{U}_q$ uses lim sup, while the other quantities are defined using the lim inf. We now state the upper estimate.

Proposition 3.6 *1) For each $q < 0$, $\dim_H \bar{L}_q \leq \dim_H L_q \leq \bar{f}(\alpha(q))$.*
2) For each $q \geq 0$, $\dim_H \bar{U}_q \leq \dim_H U_q \leq \bar{f}(\alpha(q))$.

Proof. We first prove 1). Using Proposition B.4, we have

$$-\frac{\log p_t(x,x)}{\log t} \leq \frac{\sum_{i=1}^n \log(1/\mu_{w_i})}{\sum_{i=1}^n (\log(1/\mu_{w_i}) + \log \rho_{w_i})} - \frac{c_{B.1}}{\log t},$$

for $x = F_{w|n}(\bar{x}_n) \in K \setminus V_*$ and $t_{w|n}^{-1} < t < t_{w|(n-1)}^{-1}$, where we use $t_i = \rho_i/\mu_i$ ($i \in S$). From this estimate, and the fact that $\gamma(\alpha)$ is monotone increasing on $\mathbf{R}_+$, we have $\bar{L}_q \subset L_q$ and the first inequality holds. The second inequality can be proved in the same way as Theorem 2.5 in [5].

For 2), using Theorem A.1 1), we have

$$\begin{aligned}\limsup_{t\to 0} -\frac{\log p_t(x,x)}{\log t} &\geq \limsup_{n\to\infty} \frac{1}{-\frac{\log c_{A.1}}{n} + 1 + \frac{c_{A.2}p_n(x)}{n}} \frac{1}{n} \sum_{i=1}^{\Lambda_n(x)} \log(1/\mu_{w_i}) \\ &\geq \left(\limsup_{n\to\infty} \frac{1}{1 + \frac{c_{A.2}p_n(x)}{n}}\right) \left(\liminf_{k\to\infty} \frac{\sum_{i=1}^k \log(1/\mu_{w_i})}{\sum_{i=1}^k \log t_{w_i}}\right),\end{aligned}$$

where $x = F_{w|n}(\bar{x}_n) \in K \setminus V_*$, $w|n \in \Lambda_n$ for each $n \in \mathbf{N}$. See Appendix A for the definition of Λ_n, $\Lambda_n(x)$ and $p_n(x)$. The second inequality holds because, if $\Lambda_n(x) = k$, then $\sum_{i=1}^{k-1} \log t_{w_i} \leq n < \sum_{i=1}^k \log t_{w_i}$. Using $(A.1)$, one can take a subsequence $\{n_i\}_i$ such that $p_{n_i}(x) \leq L$. Thus $\limsup_{n\to\infty} 1/(1 + c_{A.2}p_n(x)/n) = 1$. From this, and the fact that $\gamma(\alpha)$ is monotone increasing on $\mathbf{R}_+$, we have $\bar{U}_q \subset U_q$ and the first inequality holds. The second inequality can be proved in the same way as Theorem 2.3 in [5] (Theorem 2.3 in [5] is stated for the lim sup, but with a simple modification, the same conclusion holds for the lim inf). ∎

Proof of Theorem 3.4. For the first part, we obtain the result for $\alpha \in (\alpha_{\min}, \alpha_{\max})$ by combining Proposition 3.5, Proposition 3.6 and the fact that V_* is a countable set. The case $\alpha \notin (\alpha_{\min}, \alpha_{\max})$ can be done similarly. The second part is a simple consequence of the fact that g is a transform of f. ∎

Remark 3.7 When K is $[0,1]^d$ $(d > 1)$ or the Sierpinski carpet, (which are not finitely ramified, thus not p.c.f. self-similar sets), we have the following partial result. For simplicity, we take $K = [0,1]^d$ (the carpet case is similar, using the results in [4], [3]). K is a self-similar set in the sense of Definition 2.1 with $\{F_i\}_{i\in S}$, $S = \{1, 2, \cdots, 2^d\}$, $V_0 = \partial[0,1]^d$ where the F_i's are affine contractions on $\mathbf{R}^d$ with the contraction rate $1/2$. Let $\mathcal{E}$ be the Dirichlet integral and $\mathcal{F}$ the usual Sobolev space on $\mathbf{L}^2(K, m)$ (m is the Lebesgue measure). Then $(\mathcal{E}, \mathcal{F})$ is the Dirichlet form on $\mathbf{L}^2(K, m)$ which corresponds to Brownian motion. This Dirichlet form has the self-similarity property (2.2) with $\rho_i \equiv \rho_K = 2^{2-d}$ for all $i \in S$. Let μ be a self-similar measure on K with $\mu(F_i(K)) = \mu_i > 0$ for $i \in S$ where μ_i satisfies $\sum_i \mu_i = 1$. In [4] it is proved that, when $\mu_i < \rho_K$ for all $i \in S$, $(\mathcal{E}, \tilde{\mathcal{F}})$ can be a local regular Dirichlet form on $\mathbf{L}^2(K, \mu)$ where $\tilde{\mathcal{F}}$ is naturally defined from $\mathcal{F}$. Note that changing the measure for the Dirichlet form (to a mutually singular one) corresponds to a (singular) time change for the associated diffusion process via some positive continuous additive functional (see Section $5, 6$ of [9] for details). It is also proved in [4] that Theorem A.1, Theorem A.2 and Theorem 2.4 hold in this setting.

Instead of the resistance metric (which we do not have in the current setting), we use the Euclidean metric. Accordingly, we must modify the definitions slightly. Define $\beta = \beta(q)$ as the (unique) real number such that $\sum_{i=1}^m \mu_i^q 2^{-\beta(q)} = 1$, and let $\alpha_{\min} = \min_i \log(1/\mu_i)$, $\alpha_{\max} = \max_i \log(1/\mu_i)$. Set f, $\bar{f}$ and J_γ as in $(3.2), (3.9)$ and (3.8). Let $a_K = \log \rho_K / \log 2$ unless $\rho_K = 1$ (when we work with the resistance metric, $a_K = 1$) and define $\tilde{g}(\gamma) = \bar{f}(a_K \gamma/(2-\gamma))$. When $\rho_K = 1$, we define $\tilde{g}(\gamma) = (\dim_H K)\delta_2(\gamma)$. We then have the following result by the same proof as that of Proposition 3.5.

$$\dim_H J_\gamma \geq \tilde{g}(\gamma) \quad \text{for} \quad \gamma \in \mathbf{R}_+ \setminus \{\frac{2\alpha_{\min}}{\alpha_{\min} + a_K}, \frac{2\alpha_{\max}}{\alpha_{\max} + a_K}\}.$$

We cannot obtain the upper estimate for $\dim_H J_\gamma$ because we do not have precise enough heat kernel estimates on V_*, which is not a countable set. (Note that

V_* is a set of points where at least one of the components is a binary rational.) For the same reason, we do not have an estimate for $\gamma \in \{\frac{2\alpha_{\min}}{\alpha_{\min}+a_K}, \frac{2\alpha_{\max}}{\alpha_{\max}+a_K}\}$.

4 Multifractal structure of the local walk dimension

In this section, we will provide a multifractal formalism for the local walk dimension. Let $\{X_t^x\}_{t\geq 0, x\in K}$ be the diffusion process such that $X_0^x = x$, which corresponds to a Dirichlet form on the p.c.f. self-similar set with a regular harmonic structure. Let

$$W_\gamma = \{x \in K : \lim_{\epsilon\to 0} \log E^x T_{B(x,\epsilon)} / \log \epsilon = \gamma\},$$

where $T_A = T_A(X_t^x) = \inf\{t \geq 0 : X_t^x \in A^c\}$ for each Borel set A and the metric is the resistance metric. Our main result in this section is the following.

Theorem 4.1 *1. Let K be a p.c.f. self-similar fractal with a regular harmonic structure, then*

$$\dim_H W_\gamma = \bar{f}(\gamma - 1) \quad \textit{for} \quad \gamma \geq 1.$$

2. Let $\beta(q)$ be defined as the number such that

$$\sum_i \mu_i^q \rho_i^{-\beta(q)} = 1.$$

The multifractal spectrum is given by

$$h(\gamma) = \inf\{\beta(q) + (\gamma - 1)q\} \quad \textit{for} \quad \gamma \in [\alpha_{\min} + 1, \alpha_{\max} + 1].$$

The proof is basically similar to those in Section 3 and we just sketch it.
In the following, we use the notation in Appendix A. Firstly, we will give estimates for the exit time from the neighbourhood of a point. They can be proved in the same way as Lemma 4.1 in [4] and Lemma 3.5 in [12].

Lemma 4.2 *There exists $c_{4.1}, c_{4.2} > 0$ such that*

$$E^x[T_{\partial D_{\Lambda_r}(x)}] \leq c_{4.1} e^{-r} \quad \textit{for} \quad x \in K,\ r > 0, \tag{4.1}$$

$$E^x[T_{\partial D_{\Lambda_r}(x)}] \geq c_{4.2} e^{-(r+l)} \quad \textit{for} \quad x \in K \setminus (\bigcup_{x_i \in \partial D_{\Lambda_r}(x)} D_{\Lambda_{r+l}}(x_i)),\ r, l > 0. \tag{4.2}$$

Lemma 4.3
$\nu(\{x=\pi(w)\in K:\lim_{\epsilon\to 0}\frac{\log E^x T_{B(x,\epsilon)}}{\log\epsilon}=\lim_{n\to\infty}\frac{\log(\mu(K_{w|n}))^{-1}}{\log\rho_{w|n}}+1\})=1.$

Proof. First, note that by a Borel-Cantelli argument, we have $\limsup_{r\to\infty}\frac{p_r(x)}{\log r}\le c_1$ for $\nu-$ a.e. x, by choosing $c_1>0$ large enough (this can be proved in a similar way to Proposition 3.3 of [4]). Thus,

$$\lim_{r\to\infty}\frac{p_r(x)}{r}=0 \qquad \nu-\text{a.e. } x. \tag{4.3}$$

Now, using (4.2) and (4.3), the following can be easily deduced (the proof is similar to that of Theorem 3.1 of [5]),

$$\limsup_{\epsilon\to 0}\frac{\log E^x T_{B(x,\epsilon)}}{\log\epsilon}\le\limsup_{n\to\infty}\frac{\log(\mu(K_{w|n}))^{-1}}{\log\rho_{w|n}}+1, \tag{4.4}$$

for all $x=\pi(w)\in K$. On the other hand, we have by the definition of $p_r(x)$,

$$B(x,\rho_\sigma^{-1}\rho_*^{p_r(x)})\subset D_{\Lambda_r}(x) \quad \text{for} \quad x\in K\setminus V_*, \tag{4.5}$$

where $\sigma\in\Lambda_r$ is chosen so that $K_\sigma=D_{\Lambda_r}(x)$ and $\rho_*\equiv\min_i\rho_i^{-1}<1$. Thus, we have $c_{4.1}e^{-r}\ge E^x T_{B(x,\epsilon')}$ using (4.1) where $\epsilon'\equiv\rho_\sigma^{-1}\rho_*^{p_r(x)}$. As $c_2 t_\sigma^{-1}\ge e^{-r}>t_\sigma^{-1}$, we obtain

$$\frac{\log E^x T_{B(x,\epsilon')}}{\log\epsilon'}\ge\frac{-r+\log c_{4.1}}{\log\rho_\sigma^{-1}+p_r(x)\log\rho_*}\ge\frac{1+\frac{\log\mu_\sigma+\log c_2}{\log\rho_\sigma^{-1}}+\frac{\log c_{4.1}}{\log\rho_\sigma^{-1}}}{1+\frac{p_r(x)}{\log\rho_\sigma^{-1}}\log\rho_*}. \tag{4.6}$$

For each ϵ small enough, take $r\in\mathbf{N}$ so that $c_3\rho_\sigma^{-1}\rho_*^{c_2\log r}<\epsilon\le\rho_\sigma^{-1}\rho_*^{c_2\log r}$ with some small $c_3>0$ where $\sigma\in\Lambda_r(x)$. By the definition of ϵ' and (4.3), we have $B(x,\epsilon)\subset B(x,\epsilon')$ and $\log\epsilon'/\log\epsilon\to 1$ as $\epsilon\to 0$ for ν-a.e. x. Thus, using (4.6), (4.3) and the above facts, we have

$$\liminf_{\epsilon\to 0}\frac{\log E^x T_{B(x,\epsilon)}}{\log\epsilon}\ge\liminf_{|\sigma|\to\infty}(\frac{\log\mu_\sigma}{\log\rho_\sigma^{-1}}+1) \quad \nu-\text{a.e. } x=\pi(\sigma)\in K. \tag{4.7}$$

Combining (4.4), (4.7) and the fact $\nu(\hat{E}_\alpha)=1$, we obtain the result. ∎

Sketch of the proof of Theorem 4.1. Using Lemma 4.3, the proof of $\dim_H W_\gamma\ge\bar{f}(\gamma-1)$ is similar to that of Proposition 3.5. Using (4.4) and (4.6), the proof of $\dim_H W_\gamma\le\bar{f}(\gamma-1)$ is similar to that of Proposition 3.6. ∎

A Appendix A

In this appendix, we will state the estimates on $p_t(x,x)$ obtained in [4]. Let Λ_n be defined by

$$\Lambda_n = \{w = w_1 \cdots w_k \in \cup_{i\geq 0} S^i : t_{w_1} \cdots t_{w_{k-1}} \leq e^n < t_{w_1} \cdots t_{w_k}\}.$$

As before, for $w = w_1 \cdots w_k$, we abbreviate $t_w = \Pi_{i=1}^k t_{w_i}, \rho_w = \Pi_{i=1}^k \rho_{w_i}$ etc. Set $V_{\Lambda_n} = \pi(\cup_{w\in\Lambda_n} wP)$. For $x \in K$ and $l \geq 0$, let

$$D_{\Lambda_l}(x) = \{C : C \text{ is a } \Lambda_l\text{-complex which contains } x\},$$

where a Λ_l-complex is defined to be a set of the form $F_w(K)$, $w \in \Lambda_l$, and $\partial D_{\Lambda_l}(x) = cl(K \setminus D_{\Lambda_l}(x)) \cap D_{\Lambda_l}(x)$. Now, for $x \in K \setminus V_*$, define

$$p_r(x) = \inf\{k : x \notin D_{\Lambda_{r(x)+k}}(\partial D_{\Lambda_r}(x))\},$$

where $\Lambda_r(x)$ is a length of the word of the Λ_r-complex to which x belongs. Since $\cap_k D_k(A) = A$ if $A \subset K$ is closed, we have $p_r(x) < \infty$. For the Sierpinski gasket, $p_r(x) = \inf\{k : x_{\Lambda_r(x)+1} \neq x_{\Lambda_r(x)+k}\}$. Let $L = \max\{l : e > (\min_{k\in S} t_k)^l\}$. Then $p_r(x)$ has the following property.

$$\text{If} \quad p_r(x) > L + 1, \quad \text{then} \quad p_{r+1}(x) < p_r(x). \tag{A.1}$$

Theorem A.1 ([4]) *There exists a jointly continuous heat kernel $p_t(x,y)$ w.r.t. $(\mathcal{E},\mathcal{F})$ and also constants $c_{A.1},\ldots,c_{A.7} > 0$ such that the following holds.*
1) (Lower estimate) For each $x \in K \setminus V_*$ and $t \leq c_{A.1}e^{-n'}$ $(n' = n + c_{A.2}p_n(x))$,

$$c_{A.3}\{\mu(D_{\Lambda_n}(x))\}^{-1} \leq p_t(x,x).$$

2) (Upper estimate) For each $x \in K\setminus V_*$ and for each t which satisfies $c_{A.4}e^{-n''} \leq t \leq c_{A.5}e^{-m'-c_{A.6}\log n'}$ $(n' = n + c_{A.2}p_n(x),\ m' = m + c_{A.2}p_m(x),\ n'' = n' + c_{A.6}\log n')$ for some $m \leq n$,

$$p_t(x,x) \leq c_{A.7}\{\min_{\substack{w\in\Lambda_{n''} \\ F_w(K)\subset D_{\Lambda_m}(x)}} \mu_w\}^{-1}.$$

Now, as in Section 2, let $\hat{\mu}$ be any Bernoulli measure on K so that $\hat{\mu}(F_i(K)) = \hat{\mu}_i$ $(i \in S)$ and $\hat{\mu}(V_0) = 0$ where $\hat{\mu}_i \geq 0$ satisfies $\sum_i \hat{\mu}_i = 1$.

Theorem A.2 ([4]) *For any $\hat{\mu}$, there exist constants $c_{A.8}, c_{A.9}, c_{A.10} > 0$ and $h : K \to [0,1]$ such that*
1) $h(x) > 0 \qquad \hat{\mu} -$ a.e.,
2) For $t < h(x)$, if $e^{-(n+1)} \leq t \leq e^{-n}$, then

$$c_{A.9}(\log \frac{1}{t})^{-c_{A.8}} \mu(D_{\Lambda_n}(x))^{-1} \leq p_t(x,x) \leq c_{A.10}(\log \frac{1}{t})^{c_{A.8}} \mu(D_{\Lambda_n}(x))^{-1}. \tag{A.2}$$

Theorem 2.4 is now a corollary of this theorem.

B Appendix B

In this appendix, we will introduce a simpler proof of the on-diagonal upper estimate for the heat kernel of the Dirichlet form on the p.c.f. self-similar set with regular harmonic structure. The method is to use the self-similarity of the reproducing kernel. Moreover, this estimate is sharper than that obtained in Theorem A.1 2). Such argument is based on the Dirichlet-Neumann bracketing which was used in [8] to obtain the asymptotic behaviour of the eigenvalue counting function.

For $f \in l(K)$, define $\sigma_i : l(K) \to l(K) \quad (1 \leq i \leq N)$ by

$$\sigma_i f(\omega) = f \circ F_i(\omega) \quad \text{for} \quad \omega \in K.$$

Definition B.1 *For $1 \leq i \leq N$, we define the following.*
1) $\mathcal{E}^i_m(f,g)$ is a symmetric form on $l(F_i(V_m))$ defined by

$$\mathcal{E}^i_m(f,g) = \rho_i \mathcal{E}_m(\sigma_i f, \sigma_i g).$$

2) We define a subspace $\mathcal{F}_i \subset l(F_i(V_*))$ by

$$\mathcal{F}_i = \{f | f \in l(F_i(V_*)), \quad \lim_{m\to\infty} \mathcal{E}^i_m(f,f) < \infty\},$$

and a symmetric form $\mathcal{E}^i$ on $\mathcal{F}_i$ by

$$\mathcal{E}^i(f,g) = \lim_{m\to\infty} \mathcal{E}^i_m(f,g) \quad \textit{for} \quad f,g \in \mathcal{F}_i.$$

By definition, $\sigma_i \mathcal{F}_i = \mathcal{F}$. Now, let

$$\begin{aligned}
\mathcal{F}_0 &= \{f \in \mathcal{F} : f|_{V_0} = 0\}, \\
\hat{\mathcal{F}} &= \{f \in \mathbf{L}^2(K,\mu) : \exists f_i \in \mathcal{F}_i, f|_{K_i \setminus V_1} = f_i \text{ (for all } i \in S)\}, \\
\hat{\mathcal{F}}_0 &= \{f \in \mathcal{F} : f|_{V_1} = 0\}.
\end{aligned}$$

Set $\hat{\mathcal{E}}(f,g) = \sum_i \mathcal{E}^i(f_i, g_i)$ for $f, g \in \hat{\mathcal{F}}$. Then, $(\hat{\mathcal{E}}, \hat{\mathcal{F}})$ is a regular local Dirichlet form on $\mathbf{L}^2(K,\mu)$. Note that

$$\hat{\mathcal{F}}_0 \subset \mathcal{F}_0 \subset \mathcal{F} \subset \hat{\mathcal{F}}. \tag{B.1}$$

Let g_λ^0, g_λ be the λ-order reproducing kernels for $(\mathcal{E}, \mathcal{F}_0), (\mathcal{E}, \mathcal{F})$ respectively. Also, let $\hat{g}_\lambda^0, \hat{g}_\lambda$ be the λ-order reproducing kernels for $(\mathcal{E}, \hat{\mathcal{F}}_0), (\hat{\mathcal{E}}, \hat{\mathcal{F}})$ respectively. Note that as the harmonic structure is regular, the former kernels are continuous on $K \times K$ and the latter are continuous on $\cup_{i \in S} K_i \times K_i$.

Lemma B.2 *For each $x \in K \setminus V_1$,*

$$\hat{g}_\lambda^0(x,x) \le g_\lambda^0(x,x) \le g_\lambda(x,x) \le \hat{g}_\lambda(x,x).$$

Proof. Noting that
$$\frac{1}{g_\lambda(x,x)} = \inf_{u \in \mathcal{L}_x} \mathcal{E}_\lambda(u,u),$$
where $\mathcal{L}_x = \{u \in \mathcal{F} : u(x) \ge 1\}$ (similar formulae also hold for $\hat{g}_\lambda^0, g_\lambda^0, \hat{g}_\lambda$), we obtain the result using $(B.1)$. ∎

Lemma B.3 *For $x \in K \setminus V_1$, we have*

$$\begin{aligned}
g_\lambda^0(x,x) &= \rho_i \hat{g}_{t_i\lambda}^0(F_i(x), F_i(x)), \\
g_\lambda(x,x) &= \rho_i \hat{g}_{t_i\lambda}(F_i(x), F_i(x)).
\end{aligned}$$

Proof. This can be proved in the same way as [7] Proposition 2.8. ∎

Iterating Lemma $B.2$ using Lemma $B.3$, we have for $w|n \in S^n$ and $\bar{x}_n \in K \setminus V_0$,

$$\begin{aligned}
g_\lambda^0(\bar{x}_n, \bar{x}_n) &\le \rho_{w|n} g^0_{t_{w|n}\lambda}(F_{w|n}(\bar{x}_n), F_{w|n}(\bar{x}_n)) \\
&\le \rho_{w|n} g_{t_{w|n}\lambda}(F_{w|n}(\bar{x}_n), F_{w|n}(\bar{x}_n)) \le g_\lambda(\bar{x}_n, \bar{x}_n).
\end{aligned} \tag{B.2}$$

On the other hand, for $x \in K \setminus V_*$ and $\lambda > 1$, take n so that $t_{w|(n-1)} \le \lambda < t_{w|n}$ where $w|n \in S^n$ is the first n letters of the word labeling the complex which x belongs to. Taking

$$d_s^n(x) = 2\log(1/\mu_{w|n})/\log t_{w|n},$$

(thus $1 - d_s^n(x)/2 = \log \rho_{w|n}/\log t_{w|n},$) we have, by our choice of λ, that

$$c_1 \rho_{w|(n-1)} g_{t_{w|n}}(x,x) \le \lambda^{1-d_s^n(x)/2} g_\lambda(x,x) \le \rho_{w|n} g_{t_{w|(n-1)}}(x,x),$$

for some $c_1 > 0$ independent of x, n. If we write $x = F_{w|n}(\bar{x}_n)$ and apply $(B.2)$ with $\lambda = 1$ to both sides of the above inequality, we have

$$c_2 g_1^0(\bar{x}_n, \bar{x}_n) \le \lambda^{1-d_s^n(x)/2} g_\lambda(x,x) \le c_3 g_1(\bar{x}_{n-1}, \bar{x}_{n-1}) \le c_4, \qquad \text{(B.3)}$$

where $c_4 = c_3 \max_{x \in K} g_1(x,x)$.

Proposition B.4 *There exists* $c_{B.1} > 0$ *such that if* $x = F_{w|n}(\bar{x}_n) \in K \setminus V_*$ *and* $t_{w|n}^{-1} \le t < t_{w|(n-1)}^{-1}$, *then*

$$p_t(x,x) \le c_{B.1} t^{-d_s^n(x)/2}.$$

Proof. As $g_\lambda(x,x) = \int_0^\infty e^{-\lambda t} p_t(x,x)dt$, we have

$$c_4 \lambda^{d_s^n(x)/2-1} \ge g_\lambda(x,x) \ge p_a(x,x) \int_0^a e^{-\lambda t} dt = p_a(x,x) \frac{1 - e^{-\lambda a}}{\lambda},$$

where we apply $(B.3)$ for the first inequality and the monotone decreasing property of p_t in the second inequality. Taking $a = \lambda^{-1} = t$, we obtain the result. ∎

Note that the argument in this section works only for the processes that have bounded reproducing kernels. Unfortunately, the proof of the lower estimate cannot be simplified by this method (see [4], [15] for the methods of the lower estimate).

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