Random walks on graphs – lectures at RIMS

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Contents

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1. Introduction.

We start with some basic definitions. To be quite formal initially, a graph is a pair $\Gamma = (G, E)$. Here G is a finite or countably infinite set, and E is a subset of $\mathcal{P}_{1,2}(G) = \{$subsets A of G with 1 or 2 elements$\}$. Given a set A we write $|A|$ for the number of elements of A. The elements of G are called *vertices*, and the elements of E *edges*.

Now for some general definitions for graphs.

(1) We write $x \sim y$ to mean $\{x, y\} \in E$, and say that y is a *neighbour* of x. Note that since we can have $\{x\} \in E$ we have allowed edges between a point x and itself. We write $\{x, x\}$ for this edge rather than $\{x\}$.

(2) We define $d(x, y)$ to be the length n of the shortest path $x = x_0, x_1, \ldots, x_n = y$ with $x_{i-1} \sim x_i$ for $1 \le i \le n$. If there is no such path then we set $d(x, y) = \infty$. (This is the *graph* or *chemical metric* on Γ.

(3) Γ is *connected* if $d(x, y) < \infty$ for all x, y.

(4) Let
$$B(x, r) = \{y : d(x, y) \le r\}, \quad x \in G, \quad r \in [0, \infty).$$

(5) For $A \subset G$ define the *exterior boundary of* A by
$$\partial A = \partial_e A = \{y \in A^c : \text{ there exists } x \in A \text{ with } x \sim y\}.$$
Set also
$$A^o = A - \partial(A^c).$$

(6) Γ is *locally finite* if $N(x) = \{y : y \sim x\}$ is finite for each $x \in G$, – i.e. every vertex has a finite number of neighbours.

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From now on we will always assume:

$$\Gamma \text{ is locally finite and connected;}$$

We will treat *weighted graphs*.

Definition. We assume there exist weights (also called conductances) μ_{xy}, $x, y \in G$ satisfying:

(i) $\mu_{xy} = \mu_{yx}$,

(ii) $\mu_{xy} \geq 0$ all x, y,

(iii) $\mu_{xy} > 0$ if and only if $x \sim y$.

We call (Γ, μ) a *weighted graph.* The *natural weights* on Γ are given by

$$\mu_{xy} = \begin{cases} 1 & \text{if } x \sim y, \\ 0 & \text{otherwise.} \end{cases} \tag{1.1}$$

Whenever we discuss a graph without explicit mention of weights, we assume we are using the natural weights.

Let $\mu_x = \mu(x) = \sum_y \mu_{xy}$, and extend μ to a measure on A by setting

$$\mu(A) = \sum_{x \in A} \mu(x). \tag{1.2}$$

Since Γ is locally finite we have:
(i) $|B(x,r)| < \infty$ for any x and r,
(ii) $\mu(A) < \infty$ for any finite set A.
(Here $|A|$ denotes the number of elements in the set A.)

Condition (iii) above relates the weights to the graph structure. But it is not enough in some circumstances, since $0 < \mu_{xy} \ll \mu_x$ is still possible.

Definition. We say (Γ, μ) has *controlled weights* if there exists $C_2 < \infty$ such that

$$\frac{\mu_{xy}}{\mu_x} \geq \frac{1}{C_2} \qquad \text{whenever } x \sim y. \tag{1.3}$$

Controlled weights is called "the p_0 condition" (where $p_0 = 1/C_2$) in some papers.

From now on we will (except possibly in some examples) assume that (Γ, μ) has controlled weights.

Examples.
1. The Euclidean lattice $\mathbb{Z}^d$. Here $G = \mathbb{Z}^d$, and $x \sim y$ if and only if $|x - y| = 1$.
2. d-ary tree. ('Binary' if $d = 2$). This is the unique infinite graph with $\mu(x) \equiv d + 1$, and with no closed loops.
3. We will also consider the 'rooted binary tree' $\mathbb{B}$. Let $\mathbb{B}_0 = \{\partial\}$, and for $n \geq 1$ let

$\mathbb{B}_n = \{0,1\}^n$. Then the vertex set is given by $\mathbb{B} = \cup_{n=0}^{\infty} \mathbb{B}_n$. For $x = (x_1, \ldots, x_n) \in \mathbb{B}_n$ set $a(x) = (x_1, \ldots, x_{n-1})$ – we call $a(x)$ the *ancestor of* x. (We set $a(z) = \partial$ for $z \in \mathbb{B}_1$.) Then the edge set of the rooted binary tree is given by

$$E(\mathbb{B}) = \big\{\{x, a(x)\} : x \in \mathbb{B} - \mathbb{B}_0\big\}.$$

4. Let $\mathcal{G}$ be a finitely generated group. Let $\Lambda = \{g_1, \ldots g_n\}$ be a set of generators, not necessarily minimal. Write $\Lambda^* = \{g_1, \ldots g_n, g_1^{-1}, \ldots g_n^{-1}\}$. Let $G = \mathcal{G}$ and let $\{g, h\} \in E$ if and only if $g^{-1}h \in \Lambda^*$. Then $\Gamma = (G, E)$ is the Cayley graph of the group $\mathcal{G}$ with generators Λ.

$\mathbb{Z}^d$ is the Cayley graph of the group $\mathbb{Z} \oplus \ldots \oplus \mathbb{Z}$, with generators g_k, $1 \le k \le d$; here g_k has 1 in the kth place and is zero elsewhere. The ternary tree is the Cayley graph of the free group with 2 generators. Note that the same group $\mathcal{G}$ has many different Cayley graphs.

Random walks on a weighted graph.

Let $X = (X_n, n \ge 0, \mathbb{P}^x, x \in G)$ be the discrete time Markov chain on G with transition probabilities given by

$$\mathbb{P}^z(X_{n+1} = y | X_n = x) = \frac{\mu_{xy}}{\mu_x}.$$

We call X the *simple random walk* (SRW) on (Γ, μ). Let $p_n(x, y)$ be the *transition density* of X with respect to the measure μ:

$$p_n(x, y) = \frac{\mathbb{P}^x(X_n = y)}{\mu_y}.$$

Note that

$$p_1(x, y) = \frac{\mu_{xy}}{\mu_x \mu_y} = p_1(y, x),$$

and that

$$p_{n+1}(x, y) = \sum_{z \in G} p_n(x, z) p_1(z, y) \mu_z.$$

It is easy to see that $p_n(x, y) = p_n(y, x)$.

In these notes I will discuss the relation between the geometry of Γ and long run behaviour of X.

Definition. Let $A \subset G$. Set

$$T_A = \min\{n \ge 0 : X_n \in A\},$$
$$T_A^+ = \min\{n \ge 1 : X_n \in A\}.$$

Note that $T_A = \infty$ if and only if X never hits A, and that if $X_0 \notin A$ then $T_A = T_A^+$. Write $T_x = T_{\{x\}}$ and

$$\tau_A = T_{A^c} = \min\{n \ge 0 : X_n \notin A\}.$$

Recall that (Γ, μ) is recurrent if and only if

$$\mathbb{P}^x(T_x^+ < \infty) = 1,$$

and that this holds if and only if

$$\sum_{n=0}^{\infty} p_n(x,x) = \infty.$$

The *type problem* for a graph Γ is to determine whether it is transient or recurrent. A special case is given by the Cayley graphs of groups, and was sometimes called *Kesten's problem*: which groups have recurrent Cayley graphs? In this case, recall that the Cayley graph depends on both the group $\mathcal{G}$ and the set of generators Λ.

Let us start with the Euclidean lattices. Polya [P] proved the following in 1921, by a combinatorial argument.

Theorem 1.1. *$\mathbb{Z}^d$ is recurrent if $d \le 2$ and transient if $d \ge 3$.*

The advantage of Polya's argument is that it is elementary, but on the other hand it is not robust. Consider the following three situations:

(1) The SRW on the hexagonal lattice in $\mathbb{R}^2$.
(2) The SRW on a graph derived from a Penrose tiling.
(3) The graph (Γ, μ) where $\Gamma = \mathbb{Z}^d$, and the weights μ satisfy $\mu_{xy} \in [c^{-1}, c]$ if $x \sim y$.

(1) could probably be handled by Polya's method, but the details would be a bit awkward, since one has to count loops. Also it is plainly a nuisance to have to give a new argument for each new lattice. Polya's method looks hopeless for (2) or (3), since it relies on having an exact expression for $P_n(x,x)$.

The problems we will be interested in are how the geometry of Γ is related to the long run behaviour of X. As far as possible we want techniques which are 'stable' under various perturbations of the graph.

Definition. Let P be some property of a weighted graph (Γ, μ) or the SRW X on it. P is *stable under bounded perturbation of weights (weight stable)* if whenever the SRW X on (Γ, μ) satisfies P and μ' are weights on Γ such that

$$c^{-1}\mu_{xy} \le \mu'_{xy} \le c\mu_{xy}, \quad x, y \in G,$$

then the SRW X' on (Γ, μ') satisfies P. (We say the weights μ and μ' are *equivalent*.)

Definition. Let (X_i, d_i), $i = 1, 2$ be metric spaces. A map $\varphi : X_1 \to X_2$ is a *rough isometry* if there exist constants $C_1 - C_2$ such that

$$C_1^{-1}(d_1(x,y) - C_2) \le d_2(\varphi(x), \varphi(y)) \le C_1 d_1(x,y) + C_2, \tag{1.4}$$

$$\bigcup_{x \in X_1} B_{d_2}(\varphi(x), C_2) = X_2, \tag{1.5}$$

If there exists a rough isometry between two spaces they are said to be *roughly isometric.* (One can check this is an equivalence relation.)

This concept was introduced in 1985 by Kanai for manifolds – see [Kan1], [Kan2]. A rough isometry between two spaces implies that the two spaces have the same large scale structure. However, to get any useful consequences of two spaces being roughly isometric one also needs some kind of local regularity. This is usually done by only considering rough isometries within a family of spaces satisfying some fixed local regularity condition. (For example, Kanai assumed the manifolds had bounded geometry.)

Exercise. *Let $\mathcal{G}$ be a finitely generated infinite group, and Λ, Λ' be two sets of generators. Let Γ, Γ' be the associated Cayley graphs. Then Γ and Γ' are roughly isometric.*

For rough isometries of weighted graphs, the natural additional regularity is to require that both graphs have contolled weights.

Definition. Let (Γ_i, μ_i), $i = 1, 2$ be weighted graphs satisfying (1.3) (i.e. controlled weights). A map $\varphi : G_1 \to G_2$ is a *rough isometry* (between (Γ_1, μ_1) and (Γ_2, μ_2)) if
(i) φ is a rough isometry between the metric spaces (G_1, d_{Γ_1}) and (G_2, d_{Γ_2}) (with constants C_1 and C_2).
(ii) There exists $C_3 < \infty$ such that for all $x \in G_1$

$$C_3^{-1}\mu_1(x) \le \mu_2(\varphi(x)) \le C_3\mu_1(x). \tag{1.6}$$

We define *stability under rough isometries* of a property P of Γ or X in the obvious way.

Remark. Let $G = \mathbb{Z}_+$, and (for $\alpha > 0$) let $\mu^{(\alpha)}_{n,n+1} = \alpha^n$. Let $\varphi : \mathbb{Z}_+ \to \mathbb{Z}_+$ be the identity map. Then φ is (of course) a rough isometry between the metric spaces $\mathbb{Z}_+$ and $\mathbb{Z}_+$, but if $\alpha \neq \beta$ it is not a rough isometry between the weighted graphs $(\Gamma, \mu^{(\alpha)})$ and $(\Gamma, \mu^{(\beta)})$. We will see in Chapter 2 that $(\Gamma, \mu^{(\alpha)})$ is recurrent if only if $\alpha \le 1$, and that the type of a weighted graph is stable under rough isometries (of weighted graphs).

Question. Is there any 'interesting' property of a weighted graph (Γ, μ) which is weight-stable but not stable under rough isometries? (Examples of 'uninteresting' properties are Γ being bipartite or a tree.)

Definitions.

Define the function spaces

$$\begin{aligned}
C(G) &= \mathbb{R}^G = \{f : G \to \mathbb{R}\}, \\
C_0(G) &= \{f : G \to \mathbb{R} \text{ such that } f \text{ has finite support}\}, \\
C_\infty(G) &= \{f \in C(G) : \text{ for all } \varepsilon > 0 \ \{x : |f(x)| > \varepsilon\} \text{ is finite. } \}
\end{aligned}$$

The (probabilistic) Laplacian Δ is defined by

$$\Delta f(x) = \frac{1}{\mu_x} \sum_{y \in G} \mu_{xy}(f(y) - f(x)) = \mathbb{E}^x f(X_1) - f(x).$$

For $f, g \in C(G)$ define the quadratic form

$$\mathcal{E}(f,g) = \tfrac{1}{2} \sum_{x\in G} \sum_{y \in G} \mu_{xy}(f(x) - f(y))(g(x) - g(y)),$$

whenever this sum converges absolutely. This is the discrete analogue of $\int \nabla f . \nabla g$.

Note that Δ is self-adjoint on $L^2(\Gamma, \mu)$:

$$\begin{aligned}(-\Delta f, g) &= -\sum_x \Delta f(x) g(x) \mu_x = \sum_x \sum_y \mu_{xy} g(x)(f(x) - f(y)) \\ &= \tfrac{1}{2} \sum_x \sum_y \mu_{xy} g(x)(f(x) - f(y)) + \tfrac{1}{2} \sum_y \sum_x \mu_{xy} g(y)(f(y) - f(x)) \\ &= \mathcal{E}(f,g) = \mathcal{E}(g,f) = (-\Delta g, f).\end{aligned}$$

We sometimes call this the discrete Gauss-Green formula.

2. Random walks and electrical resistance.

Doyle and Snell [DS] made explicit this connection, which was implicit in many previous works, such as [BD], [NW].

Given a weighted graph $\Gamma = (G, \mu)$ one can interpret it as an electrical network: the vertices are 'nodes' and the edge $\{x, y\}$ corresponds to a wire with conductivity μ_{xy}. (We only consider 'pure resistor' networks – no impedences or capacitors.)

We fix points x_0, x_1 in G, and suppose that an external source (voltage or current) is attached at these points. Let $B = \{x_0, x_1\}$. We write I_{xy} for the current flowing from x to y; we have $I_{xy} = 0$ if $x \not\sim y$, and $I_{xy} = -I_{yx}$. Let $V(x)$ be the potential at $x \in G$.

To see how to calculate I and V (and how to define these in a mathematically precise way) we use two axioms. The first of these is Ohm's law: in an edge $\{x, y\}$ we have

$$V(y) - V(x) = I_{xy} \mu_{xy}^{-1}. \tag{2.1}$$

The second is conservation of current: at any vertex $x \in G_0 = G - \{x_0, x_1\}$ we have

$$\sum_y I_{xy} = 0. \tag{2.2}$$

By (2.1)

$$\sum_y I_{xy} = \sum_y \mu_{xy}(V(y) - V(x)) = \mu_x \Delta V(x); \tag{2.3}$$

so by (2.2) V is harmonic on G_0.

Let $V_0(x_k) = k$, $k = 0, 1$. Then V satisfies

$$\begin{aligned} V(x) &= V_0(x) \text{ on } B \\ \Delta V(x) &= 0 \quad \text{on } G - B. \end{aligned} \tag{DP}$$

Lemma 2.1. *Let* (Γ, μ) *be finite. Then (DP) has a unique solution.*

Proof. Let V_i, $i = 1, 2$ be solutions, and $u = V_1 - V_2$. Then $\Delta u(x) = 0$ for $x \in G_0$, while $u(x) = 0$ if $x \in B$. So

$$\mathcal{E}(u, u) = (-\Delta u, u) = 0,$$

and hence u is constant. Since $u(x_1) = 0$ it follows that $u = 0$. □

The connection with random walks is given by the following. Let

$$\varphi(x) = \mathbb{P}^x(T_{x_1} < T_{x_0}). \tag{2.4}$$

Then $\varphi(x_0) = 0$, $\varphi(x_1) = 1$ and if $x \in G_0$ we have by the Markov property of X

$$\varphi(x) = \mathbb{E}^x \varphi(X_1),$$

so that $\Delta\varphi(x) = 0$. Thus φ is also a solution of (DP) and so, if G is finite, we have $\varphi = V$.

Remark. If (Γ, μ) is an infinite transient graph then $h_B(x) = \mathbb{P}^x(T_B = \infty) > 0$ for some or all $x \in G - B$. So uniqueness will fail for (DP) since the function $\varphi + \lambda h_B$ is also a solution of (DP). In this case Ohm's law and conservation of current are not enough to specify the potential V and current I arising from the potential V_0 on B. We remark that if we impose the additional condition that V has 'minimal energy', then we do have uniqueness.

At this point the connection between random walks and electrical networks has done nothing to help us understand random walks better. The key concept which does make a difference is that of *effective resistance.*

Note that in the situation above (current flowing from x_0 to x_1 in a finite network) the current flowing into x_0 from the external source is, by conservation of current, the same as the flow out of x_0 into G, that is

$$F(I, x_0) = \sum_y I_{x_0 y}.$$

Definition. The *effective resistance between* x_0 *and* x_1, denoted $R_{\text{eff}}(x_0, x_1)$, is $F(I; x_0)^{-1}$, where I is the current flowing from x_0 to x_1 when x_k is held at potential k, $k = 0, 1$. It is easy to see that $R_{\text{eff}}(x_0, x_1) = R_{\text{eff}}(x_1, x_0)$.

We have (as $V = \varphi$ and $V(x_0) = 0$)

$$\begin{aligned} R_{\text{eff}}(x_0, x_1)^{-1} &= \sum_y I_{x_0 y} = \sum_y \mu_{x_0 y}(\varphi(y) - \varphi(x_0)) \\ &= \mu_{x_0} \Delta\varphi(x_0) = \mu_{x_0} \mathbb{E}^{x_0} \varphi(X_1) = \mu_{x_0} \mathbb{P}^{x_0}(T_{x_1} < T_{x_0}^+). \end{aligned} \tag{2.5}$$

We have proved:

Theorem 2.2. *If (Γ, μ) is finite then*

$$\mu_{x_0} \mathbb{P}^{x_0}(T_{x_1} < T_{x_0}^+) = \frac{1}{R_{\text{eff}}(x_0, x_1)}. \tag{2.6}$$

The following result was only discovered in 1989 – see [CRRST]. Another proof can be found in [Tet].

Theorem 2.3. *Let (Γ, μ) be a finite weighted graph. Then*

$$\mathbb{E}^{x_0} T_{x_1} + \mathbb{E}^{x_1} T_{x_0} = R_{\text{eff}}(x_0, x_1)\mu(G). \tag{2.7}$$

Proof. Write $R = R_{\text{eff}}(x_0, x_1)$. Let $\varphi_1(x) = \mathbb{P}^x(T_{x_1} < T_{x_0})$, $\varphi_0 = 1 - \varphi_1$, and $f_0(x) = \mathbb{E}^x T_{x_0}$. Then

$$(\varphi_1, -\Delta f_0) = (-\Delta\varphi_1, f_0). \tag{2.8}$$

We now calculate both sides of (2.8). We have $f_0(x_0) = 0$ and

$$\Delta f_0(x) = \mathbb{E}^x f_0(X_1) - f_0(x) = -1,$$

while $\Delta\varphi_1(x) = 0$ for $x \in G_0$, and $\varphi_1(x_0) = 0$, $\varphi_1(x_1) = 1$. Also, by (2.5) we have

$$R^{-1} = \mu_{x_0}\Delta\varphi_1(x_0) = \mu_{x_1}\Delta\varphi_1(x_1).$$

The left side of (2.8) is

$$\sum_{x \neq x_1} \varphi_1(x)\mu_x + \varphi_1(x_0)(-\Delta\varphi_1(x_0))\mu_{x_0} = \sum_x \varphi_1(x)\mu_x,$$

and the right side is

$$-\Delta\varphi_1(x_0) f_0(x_0)\mu_{x_0} + -\Delta\varphi_1(x_1) f_0(x_1)\mu_{x_1} = R^{-1} f_0(x_1).$$

So,

$$f_0(x_1) = \mathbb{E}^{x_1} T_{x_0} = R_{\text{eff}}(x_0, x_1) \sum_x \varphi_1(x)\mu_x. \tag{2.9}$$

Similarly if $f_1(x) = \mathbb{E}^x T_{x_1}$, then

$$\mathbb{E}^{x_0} T_{x_1} = R_{\text{eff}}(x_0, x_1) \sum_x \varphi_0(x)\mu_x.$$

Since $\varphi_0(x) + \varphi_1(x) = 1$, adding these two equations completes the proof. □

Definition. Let $A \subset G$. The *graph obtained by collapsing A to a point a* is the graph $\Gamma' = (G', E')$ defined as follows. We have $G' = (G - A) \cup \{a\}$. Let $B = \{x : x \sim b$ for some $b \in A\}$; then

$$E' = \big\{\{x, y\} : x, y \in G - A\big\} \cup \big\{\{x, a\} : x \in B\big\}.$$

We define weights on (G', E') by setting $\mu'_{xy} = \mu_{xy}$ if $x, y \in G - A$, and

$$\mu'_{xa} = \sum_{b \in A} \mu_{xb}.$$

Note that $\mu'_{xa} \le \mu_x < \infty$. The graph Γ' need not be locally finite, since the new point a might have infinitely many neighbours, but will be if either A or $G - A$ is finite.

Corollary 2.4. *(See [T1]. Let (Γ, μ) be a graph, and $A \subset G$ be finite. Then*

$$\mathbb{E}^x \tau_A \le R_{\rm eff}(x_0, A^c)\mu(A).$$

Proof. This follows from (2.9) on collapsing A^c to a single point x_0. Note that as $\varphi_1(x_0) = 0$ we obtain on the right hand side

$$\sum_{x \neq x_0, x\in A} \varphi_1(x)\mu_x \le \mu(G - A).$$

□

Remarks. 1. Note that in an unweighted graph $\mu(G) = 2|E|$.
2. Let $x, y, z \in G$. Then

$$\mathbb{E}^x T_y + \mathbb{E}^y T_z \ge \mathbb{E}^x T_z,$$
$$\mathbb{E}^y T_x + \mathbb{E}^z T_y \ge \mathbb{E}^z T_x;$$

adding one deduces that $R_{\rm eff}(x, y) + R_{\rm eff}(y, z) \ge R_{\rm eff}(x, z)$. Thus $R_{\rm eff}(.,.)$ defines a metric ('the resistance metric') on G. (See Theorem 1.6 in [Ki].)

Example. Suppose (Γ, μ) is a transient weighted graph, $\{x_0, x_1\}$ is an edge in Γ, and there is a finite subgraph $G_1 = \{x_1, \dots, x_n\} \subset G$ 'hanging' from x_1. More precisely, if $y \in G_1$ then every path from y to x_0 passes through x_1. Then we can ask how much the existence of G_1 'delays' the SRW X, that is, what is $E^{x_1} T_{x_0}$?

Let $G_0 = \{x_0, \dots x_n\}$, and work with the subgraph generated by G_0, except we eliminate any edges between x_0 and itself. Write μ^0 for the measure on G_0 given by (1.2), and for clarity denote the hitting times for the SRW on G_0 by T^0_y. Then, by Theorem 2.3,

$$E^{x_0} T^0_{x_1} + E^{x_1} T^0_{x_0} = R_{\rm eff}(x_0, x_1)\mu^0(G_0).$$

Since $E^{x_0} T^0_{x_1} = 1$, $R_{\rm eff}(x_0, x_1) = \mu^{-1}_{x_0,x_1}$, $\mu_0(G_0) = \mu(G_1) + \mu^{-1}_{x_0,x_1}$, and $E^{x_1} T^0_{x_0} = E^{x_1} T_{x_0}$, we obtain

$$E^{x_1} T_{x_0} = \frac{\mu(G_1)}{\mu_{x_0,x_1}}.$$

Transience and Recurrence

We can extend the definition of effective resistance from pair of points to pairs of sets in a straightforward fashion. Let G be a graph (finite or infinite), B_i be disjoint sets, and suppose $G_0 = G - (B_0 \cup B_1)$ is finite. Then $R_{\rm eff}(B_0, B_1)$ is the same as $R_{\rm eff}(x_0, x_1)$ in the graph Γ' obtained by collapsing the sets B_i to single points x_i. (A sufficient condition for Γ' to be locally finite is that at least one of B_0, B_1 is finite.)

Let (Γ, μ) be an infinite weighted graph, and let (A_n) be finite with $A_n \uparrow G$ – i.e. $A_n \subset A_{n+1}$, $\cup A_n = G$. Assume that $B(x_0, 1) \subset A_1$. We define

$$R_{\rm eff}(x_0, \infty) = \lim_n R_{\rm eff}(x_0, A^c_n). \tag{2.10}$$

Lemma 2.5. *(a) If $A_1 \subset A_2$ then $R_{\text{eff}}(x_0, A_1^c) \le R_{\text{eff}}(x_0, A_2^c)$.*
(b) $R_{\text{eff}}(x_0, \infty)$ does not depend on the sequence (A_n) used.

Proof. As $\tau_{A_1} \le \tau_{A_2}$ (a) is immediate from Theorem 2.2. (We will see a better proof later.)
(b) Let A'_n be another sequence, and denote the limits in (2.10) by R and R'. Given n there exists m_n such that $A_n \subset A'_{m_n}$. So, by (a),

$$R_{\text{eff}}(x_0, A_n^c) \le R_{\text{eff}}(x_0, (A'_{m_n})^c) \le \lim_m R(x_0, (A'_m)^c) = R'.$$

Thus $R \le R'$ and similarly $R' \le R$. □

Theorem 2.6. *Let (Γ, μ) be an infinite weighted graph. For $x_0 \in G$*

$$\mu_{x_0} \mathbb{P}^{x_0}(T_{x_0}^+ = \infty) = \frac{1}{R_{\text{eff}}(x_0, \infty)}. \tag{2.11}$$

Proof. Let $A_n = B(x_0, n)$. Let G_n be the finite graph obtained by collapsing A_n^c to a single point x_1. Then by Theorem 2.2 $\mu_{x_0} \mathbb{P}^{x_0}(T_{A_n^c} < T_{x_0}^+) = R_{\text{eff}}(x_0, A_n^c)^{-1}$. Letting $n \to \infty$ gives (2.11). □

Corollary 2.7. *(a) (Γ, μ) is recurrent if and only if $R_{\text{eff}}(z_1, \infty) = \infty$.*
(b) (Γ, μ) is transient if and only if $R_{\text{eff}}(z_1, \infty) < \infty$.

Examples. 1. $\mathbb{Z}$ is recurrent since $R_{\text{eff}}(\{0\}, [-n, n]^c) = \frac{1}{2}(n+1)$.
2. For $\mathbb{Z}^2$ one reduces effective resistance if one 'shorts' boxes around 0. (See *** below for a proof of this.) Then

$$R_{\text{eff}}(0, \infty) = \frac{1}{4} + \frac{1}{12} + \frac{1}{20} + \frac{1}{28} + \ldots = \frac{1}{4}(1 + \frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \ldots) = \infty.$$

2. Consider the graph (Γ, μ) mentioned in Lecture 1, with vertex set $\mathbb{Z}_+$ and weights $\mu_{n,n+1} = \alpha^n$. Then $R_{\text{eff}}(0, n) = \sum_{r=0}^{n-1} \alpha^{-r}$, so that (Γ, μ) is transient if and only if $\alpha > 1$.

Energy and Variational Methods

In the examples I used various methods of calculating or bounding effective resistance techniques without any justification. But as facts like "shorts decrease resistance" should be provable from the definition, I'll now go on to discuss in more detail the mathematical background to effective resistance.

Let (Γ, μ) be a finite weighted graph, and $B_0, B_1 \subset G$ with $B_0 \cap B_1 = \emptyset$. Write $B = B_0 \cup B_1$, $G_0 = G - B$. We now introduce two variational problems. For the first, set

$$\mathcal{F}(B_0, B_1) = \{f \in H^2(G) : f|_{B_1} = 1, f|_{B_0} = 0\},$$

and consider the energy minimising problem:

$$C_{\text{eff}}(B_0, B_1) = \min\{\mathcal{E}(f, f) : f \in \mathcal{F}(B_0, B_1)\}. \tag{$VP1$}$$

We call $C_{\text{eff}}(B_0, B_1)$ the 'effective conductance' between B_0 and B_1, and will (of course) find that it is the reciprocal of $R_{\text{eff}}(B_0, B_1)$.

The second problem is to minimise the energy of flows from B_0 to B_1. We call $J = (J_{xy})$ a *flow from B_0 to B_1* if
(1) $\sum_y J_{xy} = 0$ if $x \in G_0$,
(2) $J_{xy} = -J_{yx}$,
(3) $J_{xy} = 0$ if $x \not\sim y$.

We call the *flux* of J the total amount of flow out of B_0, that is

$$F(J, B_0) = \sum_{x \in B_0} \sum_y J_{xy}.$$

Since $J_{xy} = -J_{yx}$ and $\sum_y J_{xy} = 0$ if $x \notin B$ we have

$$0 = \sum_x \sum_y J_{xy} = \sum_{x \in B_0} \sum_y J_{xy} + \sum_{x \in B_1} \sum_y J_{xy} = F(J, B_0) + F(J, B_1),$$

giving $F(J, B_1) = -F(J, B_0)$. Given flows I, J let

$$E(I, J) = \tfrac{1}{2} \sum_x \sum_{y \sim x} I_{xy} J_{xy} \mu_{xy}^{-1};$$

thus $E(I, I)$ is the energy dissipation of the flow I.

The second variational problem is:

$$R(B_0, B_1) = \inf\{E(J, J) : \; J \text{ is a flow from } B_0 \text{ to } B_1 \text{ with } F(B_0, J) = 1 \}. \qquad (VP2)$$

Theorem 2.8. *In a finite graph $C_{\text{eff}}(B_0, B_1)^{-1} = R_{\text{eff}}(B_0, B_1) = R(B_0, B_1)$.*

Proof. Let $\varphi(x) = \mathbb{P}^x(T_{B_1} < T_{B_0})$, and let $I_{xy} = \mu_{xy}(\varphi(y) - \varphi(x))$ be the flow due to φ. Then

$$\begin{aligned} R_{\text{eff}}(B_0, B_1)^{-1} = F(I, B_0) &= \sum_{x \in B_0} \sum_y I_{xy} \\ &= \sum_{x \in B_0} \sum_y \mu_{xy}(\varphi(y) - \varphi(x)) = \sum_{x \in B_0} \mu_x \Delta \varphi(x). \end{aligned}$$

Also,

$$\mathcal{E}(\varphi, \varphi) = \mathcal{E}(1 - \varphi, 1 - \varphi) = (-\Delta(1 - \varphi), 1 - \varphi) = (\Delta \varphi, 1 - \varphi) = \sum_{x \in B_0} \Delta \varphi(x) \mu_x.$$

So we deduce that $\mathcal{E}(\varphi, \varphi) = F(I, B_0) = R_{\text{eff}}(B_0, B_1)^{-1}$.

Now let f be a feasible function for (VP1). So $f_{B_k} = k$, $k = 0, 1$. Then

$$\mathcal{E}(\varphi, f - \varphi) = (-\Delta \varphi, f - \varphi) = 0,$$

so $\mathcal{E}(\varphi, f) = \mathcal{E}(\varphi, \varphi)$. Hence

$$0 \le \mathcal{E}(f - \varphi, f - \varphi) = \mathcal{E}(f, f) - \mathcal{E}(\varphi, \varphi).$$

It follows that φ is the unique minimiser for (VP1).

To handle (VP2) we first prove an identity. Let J be any flow feasible for (VP2), and let f be feasible for (VP1). Then

$$\begin{aligned}\tfrac{1}{2}\sum_x\sum_y J_{xy}(f(y) - f(x)) &= \tfrac{1}{2}\sum_{x,y} J_{xy} f(y) - \tfrac{1}{2}\sum_{x,y} J_{xy} f(x)\\ &= \sum_{x,y} J_{xy} f(y) = \sum_{y \in B} f(y) \sum_x J_{xy}\\ &= -F(J, B_1) = F(J, B_0) = 1.\end{aligned}$$

Then, as $I_{xy}\mu_{xy}^{-1} = \varphi(y) - \varphi(x)$,

$$E(I, J) = \tfrac{1}{2}\sum_x\sum_y J_{xy}(\varphi(y) - \varphi(x)) = 1.$$

Let $I' = R_{\text{eff}}(B_0, B_1)I$, so that I' is feasible for (VP2). Then $E(I', J) = R_{\text{eff}}(B_0, B_1)$, and taking $J = I'$ we deduce that

$$0 \le E(J - I', J - I') = E(J, J) - E(I', I').$$

Hence I' is the unique minimiser for (VP2), and $R(B_0, B_1) = R_{\text{eff}}(B_0, B_1)$.

Remarks. 1. We can still define (VP1) and (VP2) if Γ is infinite, and it is still true that $C_{\text{eff}}(B_0, B_1) = R_{\text{eff}}(B_0, B_1)^{-1}$. However, there are problems with (VP2). For example, let Γ consist of two copies of a rooted binary tree, denoted G_0, G_1, with roots x_0 and x_1, connected by a path $A = \{x_0, y_1, y_2, x_1\}$ of length 3. Then if $B_j = \{x_j\}$ it is clear that $R_{\text{eff}}(B_0, B_1) = 3$ and $C_{\text{eff}}(B_0, B_1) = \frac{1}{3}$. However, one can build a flow I (going from x_0 to infinity in G_0, coming from infinity to x_1 to infinity in G_1) with $E(I, I) = 2$.

This example suggests that to handle flows on general infinite graphs one needs to impose correct boundary behaviour at infinity. For more on this see Chapter 3 of [LyPe].

2. If G is infinite, B_1 is finite then Theorem 2.8 can be extended to the case "$B_1 = \infty$" if we impose the right boundary conditions at infinity. In the case of (VP1) these are that $f \in H_0(G)$. This is the closure in the norm $||f|| = (\mathcal{E}(f, f) + ||f||_2^2)^{1/2}$ of $C_0(G)$.

For (VP2) we just take $B_0 = \emptyset$ in the definition of a flow from B_0 to B_1, and require $F(J, B_1) = -1$.

Using Theorem 2.8 we can prove easily the validity of standard operations on resistances.

Definition. ('Shorts' and 'cuts'). Let (Γ, μ) be a weighted graph, and $e = \{x, y\}$ be an edge in Γ. The graph $(\Gamma^{(c)}, \mu^{(c)})$ obtained by *cutting the edge* e is the weighted graph with vertex set G and weights given by

$$\mu_{wz}^{(c)} = \begin{cases} \mu_{wz}, & \{w, z\} \neq \{x, y\}, \\ 0, & \{w, z\} \neq \{x, y\}. \end{cases}$$

(Thus the edges of $\Gamma^{(c)}$ are the edges of (Γ, μ) with e removed.)

The graph $(\Gamma^{(s)}, \mu^{(s)})$ obtained by *shorting the edge* e is the weighted graph obtained by identifying the vertices x and y. More precisely we take $G^{(s)} = G - \{y\}$, and

$$\mu^{(s)}_{wz} = \mu_{wz}, \quad w, z \in G^{(s)} - \{x\},$$
$$\mu^{(s)}_{wx} = \mu_{wx} + \mu_{wy}, \quad w \in G^{(s)}.$$

Corollary 2.9. *Let μ, μ' be weights on Γ.*
(a) If $\mu \le C_1 \mu'$ then

$$C_{\text{eff}}(B_0, B_1) \le C_1 C'_{\text{eff}}(B_0, B_1), \qquad (2.12)$$
$$R_{\text{eff}}(B_0, B_1) \ge C_1^{-1} R'_{\text{eff}}(B_0, B_1).$$

(b) Shorts decrease R_{eff}.
(c) Cuts increase R_{eff}.

Proof. (a) Since $\mathcal{E} \le C_1 \mathcal{E}'$, (2.12) is immediate.
(b) and (c) Fix an edge e, and write $\mathcal{E}_\lambda(f,f)$ for the energy of f in the graph (Γ, μ^λ), where $\mu^\lambda_{e'} = \mu_{e'}$ if $e' \ne e$ and $\mu^\lambda_{e'} = \lambda$. Then we have

$$\mathcal{E}^{(c)}(f,f) = \mathcal{E}_0(f,f) \qquad \mathcal{E}^{(s)}(f,f) = \lim_{\lambda\to\infty} \mathcal{E}_\lambda(f,f),$$

and the result follows from (a). □

The following reformulation of Theorem 2.6, using the langauge of Sobolev inequalities rather than resistance, is useful and connects with the ideas in Section 3.

Theorem 2.10. *Let $x_0 \in G$. (Γ, μ) is transient if and only if there exists $C_1 = C_1(x_0) < \infty$ such that*

$$|f(x_0)|^2 \le C_1 \mathcal{E}(f,f) \quad \text{for all } f \in C_0(G). \qquad (2.13)$$

Proof. Let $B_n = B(x_0, n)$. Suppose first that (Γ, μ) is transient, and let $K = R_{\text{eff}}(x_0, \infty)$, so that $K = \lim_n R_{\text{eff}}(x_0, B_n^c) < \infty$. Let $f \in C_0(G)$ with $f(x_0) = 1$; then f is feasible for (VP1) for all large n. So

$$\mathcal{E}(f,f) \ge R_{\text{eff}}(x_0, B_n^c)^{-1} \ge K^{-1} = K^{-1}|f(x_0)|^2.$$

Now suppose (2.13) holds. Then using Theorem 2.8 we have that $C_{\text{eff}}(x_0, B_n^c) \ge C_1^{-1}$ for all n, so $R_{\text{eff}}(x_0, \infty) \le C_1$. □

Remark. Using these ideas one can now prove that transience and recurrence and stable under rough isometries. Details of the argument can be found in [W].

Corollary 2.11. *The type of a Cayley graph does not depend on the choice of the (finite) set of generators.*

Proof. This is immediate from
(1) the stability of transience and recurrence under rough isometries, and
(2) the fact that two Cayley graphs of a group are roughly isometric. $\square$

3. Isoperimetric inequalities and applications to transition densities.

We continue to assume that (Γ, μ) is a locally finite weighted graph satisfying the controlled weights condition. For $A, B \subset G$ set

$$E(A, B) = \{e = \{x, y\} : x \in A, y \in B\},$$
$$\mu_E(A; B) = \sum_{x \in A} \sum_{y \in B} \mu_{xy},$$

Definition. Let $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ be increasing. (Γ, μ) satisfies a ψ - *isoperimetric inequality* (denoted $I(\psi)$) if there exists $C_0 < \infty$ such that

$$\psi(\mu(A)) \leq C_0 \mu_E(A; G - A) \quad \text{for every finite } A \subset G. \tag{$I(\psi)$}$$

For $\alpha \in (1, \infty)$ we write (I_α) for $(I(\psi))$ with $\psi(t) = t^{1-1/\alpha}$, and (I_∞) for $I(\psi)$ with $\psi(t) = t$.

If $\mu_x \geq 1$ for all x then $\mu(A)^{(\alpha-1)/\alpha} \geq \mu(A)^{(\alpha'-1)/\alpha'}$ if $\alpha \geq \alpha'$. So (I_α) implies (I_β) for any $\beta < \alpha$.

Examples.
1. The Euclidean lattice $\mathbb{Z}^d$ satisfies I_d.
2. The binary tree satisfies (I_∞) with $C_0 = 3$.
3. The Sierpinski gasket graph does not satisfy (I_α) for any $\alpha > 1$.

Now define
$$||\nabla f||_p^p = \tfrac{1}{2} \sum_x \sum_y \mu_{xy} |f(y) - f(x)|^p.$$

We will be mainly interested in the cases $p = 1$ and $p = 2$: of course $||\nabla f||_2^2 = \mathcal{E}(f, f)$. Note that if $f = 1_A$ then

$$||\nabla f||_p^p = \sum_{x \in A} \sum_{y \in A^c} \mu_{xy} = \mu_E(A; A^c). \tag{3.1}$$

Theorem 3.1. *Let $\alpha \in (1, \infty]$. The following are equivalent:*
(a) (Γ, μ) satisfies (I_α) with constant C_0.
(b) (Γ, μ) satisfies the Sobolev inequality

$$||f||_{\alpha/\alpha-1} \leq C_0 ||\nabla f||_1 \quad \text{for } f \in C_\infty(G). \tag{S^1_α}$$

Proof of (b) ⇒ (a). Given a finite subset $A \subset G$ set $f = 1_A$. Then if $\alpha \in (1, \infty)$

$$||f||_{\alpha/(\alpha-1)} = \Big(\sum_{x \in A} \mu_x)\Big)^{(\alpha-1)/\alpha} = \mu(A)^{1-1/\alpha},$$

while by (3.1) $||\nabla f||_1 = \mu_E(A; A^c)$. For $\alpha = \infty$ (S^1_α) takes the form $\|f\|_1 \le C_0 \|\nabla f\|_1$, and setting $f = 1_A$ gives $\mu(A) \le C_0 \mu_E(A, A^c)$. □

To prove that (a) implies (b), first define

$$\Lambda_t(f) = \{x : f(x) \ge t\}.$$

Lemma 3.2. *(Co-area formula). Let $f : G \to \mathbb{R}_+$. Then*

$$||\nabla f||_1 = \int_0^\infty \mu_E(\Lambda_t(f), \Lambda_t(f)^c)dt.$$

Proof. We have

$$\begin{aligned} ||\nabla f||_1 &= \sum_x \sum_y \mu_{xy}(f(y) - f(x))_+ \\ &= \sum_x \sum_y \mu_{xy} \int_0^\infty 1_{(f(y) \ge t > f(x))} dt \\ &= \int_0^\infty dt \sum_x \sum_y \mu_{xy} 1_{(f(y) \ge t > f(x))} = \int_0^\infty dt \mu_E(\Lambda_t(f), \Lambda_t(f)^c). \end{aligned}$$

□

Proof of Theorem 3.1, (b) ⇒ (a). First, note that it is enough to consider $f \ge 0$. For, if f is general, and $g = |f|$ then $||g||_p = ||f||_p$ and $||\nabla g||_1 \le ||\nabla f||_1$.

The idea of the proof is to use (I_α) on the sets $\Lambda_t(f)$. However, if one does what comes naturally one finds one needs to use Hardy's inequality:

$$\int_0^\infty pt^{p-1} f(t)dt \le \Big(\int_0^\infty pt^{p-1} f(t)dt\Big)^p.$$

We can avoid this by using a trick due to Rothaus ([Ro]).

First let $1 < \alpha < \infty$. Let $p = \alpha/(\alpha - 1)$, and q be the conjugate index. Let $g \in L^q_+(G, \mu)$ with $||g||_q = 1$. Then

$$\begin{aligned} C_0 ||\nabla f||_1 &= \int_0^\infty C_0 \mu_E(\Lambda_t(f), \Lambda_t(f)^c)dt \ge \int_0^\infty \mu_E(\Lambda_t(f))^{1/p} dt \\ &= \int_0^\infty dt ||1_{\Lambda_t(f)}||_p \ge \int_0^\infty dt ||g 1_{\Lambda_t(f)}||_1 \\ &= \int_0^\infty dt \sum_g g(x) \mu_x 1_{\Lambda_t(f)}(x) = \sum_x g(x) \mu_x \int_0^\infty dt 1_{(f(x) \ge t)} \\ &= \sum_x f(x) g(x) \mu_x = ||fg||_1. \end{aligned}$$

So,

$$||f||_p = \sup\{||fg||_1 : ||g||_q = 1\} \le C_0 ||\nabla f||_1.$$

For $\alpha = \infty$ take $p = 1$, $g = 1$ in the calculation above. □

Theorem 3.3. *Let (Γ, μ) satisfy (I_α) with $\alpha \in (2, \infty]$. Then (Γ, μ) also satisfies:*

$$||f||_{2\alpha/(\alpha-2)} \le C_1 \mathcal{E}(f,f), \quad f \in C_\infty(G). \qquad (S^2_\alpha)$$

Proof. As before it is enough to prove this for $f \ge 0$. Let first $f \in C_0(G)$, and $2 < \alpha < \infty$. Set

$$\beta = \frac{2(\alpha-1)}{\alpha-2}, \qquad \text{so that} \quad 2\beta - 2 = \frac{2\alpha}{\alpha-2},$$

and let $g = f^\beta$. Then

$$||g||_{\alpha/(\alpha-1)} = ||f^\beta||_{\alpha/(\alpha-1)} = ||f||^\beta_{\alpha\beta/(\alpha-1)} = ||f||^\beta_{2\alpha/(\alpha-2)}.$$

If we were working with functions on $\mathbb{R}^d$ then we would have

$$c||g||_{\alpha/(\alpha-1)} \le \int |\nabla g| = \int |\nabla f^\beta| = \beta \int f^{\beta-1} |\nabla f| \qquad (3.2)$$

$$\le c(\int |\nabla f|^2)^{1/2} (\int |f|^{2\beta-2})^{1/2} = c\mathcal{E}(f,f) ||f||^{\beta-1}_{2\alpha/(\alpha-2)}. \qquad (3.3)$$

However, in a discrete space one cannot use the Leibnitz rule to obtain the final equality in (3.2). This is not a serious difficulty: if $a < b$ then

$$b^\beta - a^\beta = \int_a^b \beta t^{\beta-1} dt \le (b-a)\beta b^{\beta-1}.$$

So, for any $a, b \in \mathbb{R}$

$$|b^\beta - a^\beta| \le \beta |b-a| (a^{\beta-1} + b^{\beta-1}).$$

Hence,

$$\begin{aligned}
||\nabla g||_1 &= \tfrac{1}{2} \sum_{x,y} \mu_{xy} |f(x)^\beta - f(y)^\beta| \\
&\le \tfrac{1}{2}\beta \sum_{x,y} \mu_{xy} |f(x) - f(y)| |f(x)^{\beta-1} + f(y))^{\beta-1}| \\
&= \beta \sum_{x,y} \mu_{xy} |f(x) - f(y)| f(x)^{\beta-1} \\
&\le \beta \mathcal{E}(f,f)^{1/2} (\sum_x \mu_x |f(x)^{2\beta-2}|)^{1/2} = \beta \mathcal{E}(f,f)^{1/2} ||f||^{\beta-1}_{2\alpha/(\alpha-2)}.
\end{aligned}$$

So, by Theorem 3.1

$$||f||^\beta_{2\alpha/(\alpha-2)} = ||g||_{\alpha/(\alpha-1)} \le C_0 ||\nabla g||_1 \le c\mathcal{E}(f,f)^{1/2} ||f||^{\beta-1}_{2\alpha/(\alpha-2)}.$$

Since $f \in C_0(G)$ all the terms are finite, so

$$||f||_{2\alpha/(\alpha-2)} \leq c'\mathcal{E}(f,f)^{1/2}. \tag{3.4}$$

If $\alpha = \infty$ then we take $g(x) = f(x)^2$, and the same argument gives

$$||f||_2^2 = ||g||_1 \leq C_0||\nabla g||_1 \leq c\mathcal{E}(f,f)^{1/2}||f||_2.$$

Finally, if $f \in C_\infty(G)$ let $f_n = (f - 1/n)_+ \in C_0(G)$. Then by (3.4)

$$||f_n||^2_{2\alpha/\alpha-2} \leq c\,\mathcal{E}(f_n, f_n) \leq \mathcal{E}(f,f),$$

and letting $n \to \infty$ gives the result. □

Corollary 3.4. *Let (Γ, μ) satisfy (I_α) with $\alpha \in (2, \infty]$. Then (Γ, μ) is transient.*

Proof. Let $x_0 \in G$, $p = 2\alpha/(\alpha - 2)$. Then

$$|f(x_0)|^2 \leq \mu_{x_0}^{-2/p}||f||_p^2 \leq c\,\mathcal{E}(f,f),$$

and (Γ, μ) is transient by Theorem 2.10. □

Theorem 3.5. *Let $\alpha > 2$. The following are equivalent:*
(a) For $f \in C_\infty(G)$

$$\mathcal{E}(f,f) \geq c_S||f||^2_{2\alpha/(\alpha-2)}. \tag{3.5}$$

(b) Γ satisfies the inequality

$$\mathcal{E}(f,f) \geq c_N||f||_2^{2+4/\alpha}||f||_1^{-4/\alpha}, \qquad f \in L^1 \cap L^2. \tag{N_α}$$

Remark. (N_α) is called a 'Nash' inequality – the terminology, as well as Theorem 3.5, is due to [CKS]. Nash used inequalities of this type in his 1958 paper [N] (see the first inequality on p. 936) to obtain Hölder continuity of solutions of divergence form PDEs.

Proof. (a) imples (b) is easy. Let p, q be conjugate indices with $p^{-1} = (\alpha - 2)/(\alpha + 2)$, $q^{-1} = 4/(\alpha + 2)$. Then using Hölder's inequality and (a) we have:

$$\begin{aligned}
||f|_2^2 = \sum_x f(x)^2\mu_x &= \sum_x |f(x)|^{2\alpha/(\alpha+2)}|f(x)|^{4/(\alpha+2)}\mu_x \\
&\leq \Big(\sum_x |f(x)|^{2\alpha/(\alpha-2)}\Big)^{1/p}\Big(\sum_x |f(x)|\Big)^{1/q} \\
&= ||f||^{2\alpha/(\alpha+2)}_{2\alpha/(\alpha-2)}||f||_1^{4/(\alpha+2)} \\
&\leq c\mathcal{E}(f,f)^{\alpha/(\alpha+2)}||f||_1^{4/(\alpha+2)}.
\end{aligned}$$

Rearranging gives (N_α).
(b) implies (a) is harder, but we will not need this. $\square$

We now show how the techniques introduced above can be used to give bounds on transition densities. It is usually slightly easier to handle the density of the continuous time simple random walk (CTSRW) on (Γ, μ): while the essential ideas are the same in both discrete and continuous time contexts, discrete time introduces some extra (mainly minor) complications.

Let $Y = (Y_t, t \in [0,\infty), \mathbb{P}^x, x \in G)$ be the continuous time Markov chain on G with generator Δ. The process Y waits an `Exp(1)` time at each vertex x, and then jumps to some $y \sim x$ with the same jump probabilities as X, that is with probability $P(x,y) = \mu_{xy}/\mu_x$. Thus the process Y may be constructed from X and an independent rate 1 Poisson process N_t by setting

$$Y_t = X_{N_t},\ t \in [0,\infty). \tag{3.6}$$

Remarks.
1. In general for a continuous time Markov chain there is the possibility of 'explosion', that is that the process will reach the boundary of G in finite time. However, since Y cannot explode, since it waits an average of one time unit at each state x before jumping to the next state.
2. Given set G and a collection of bond conductivities μ_{xy} the mutual energy of functions $f, g \in C_0(G)$ is given as usual by

$$\mathcal{E}(f,g) = \tfrac{1}{2} \sum_{x\in G} \sum_{y \in G} \mu_{xy}(f(x) - f(y))(g(x) - g(y)).$$

Now let $\nu_x > 0$, so ν defines a measure on G. We can associate an operator $L = L_\nu$ to the pair $(\mathcal{E}, \nu)$ by requiring

$$\mathcal{E}(f,g) = \langle -Lf, g\rangle_\nu = -\sum_x Lf(x) g(x) \nu_x, \quad f, g \in C_0(G).$$

Using the discrete Gauss-Green formula we deduce that

$$L_\nu f(x) = \frac{\mu_x}{\nu_x} \Delta f(x) = \nu_x^{-1} \sum_y \mu_{xy}(f(y) - f(x)).$$

The process Y^ν with generator L_ν waits at x for an exponential time with mean ν_x/μ_x, and then jumps to $y \sim x$ with probability $P(x,y) = \mu_{xy}/\mu_x$.
3. In the general terminolgy of Dirichlet forms (see [FOT]), one would say that Y^ν is the Markov process associated with the Dirichlet form $\mathcal{E}$ on $L^2(\nu)$. Note that the quadratic form $\mathcal{E}$ alone is not enough to specify the process: one needs the measure ν also. Given two diferent measures ν, ν', the processes Y^ν, $Y^{\nu'}$ are time changes of each other.
4. When looking at random walks on a graph (Γ, μ) there are two natural choices of ν. The first (chosen above) is $\nu = \mu$, while the second is $\nu_x \equiv 1$. If we wish to distinguish these processes, we call them the *fixed speed* and *variable speed* continuous time simple

random walks on (Γ, μ) . In what follows we will just discuss the fixed speed CTSRW. (Note that explosion in finite time is a possibility for the variable speed CTSRW.)

We write $q_t(x,y)$ for the density of Y_t with respect to μ. So,

$$q_t(x,y) = \frac{\mathbb{P}^x(Y_t = y)}{\mu_y}.$$

Using the representation (3.6) we have

$$\mathbb{P}^x(Y_t = y) = \sum_{n=0}^{\infty} \mathbb{P}^x(X_n = y, N_t = n) = \sum_{n=0}^{\infty} p_n(x,y)\mu_y \mathbb{P}^x(N_t = n).$$

Hence

$$q_t(x,y) = \sum_{n=0}^{\infty} \frac{e^{-t}t^n}{n!} p_n(x,y),$$

giving the transition density of Y in terms of that of X.

Using this, or otherwise, it is easy to check that q_t satisfies the continuous time heat equation on (Γ, μ):

$$\frac{\partial}{\partial t} q_t(x,y) = \Delta q_t(x,y).$$

We also have the Chapman-Kolmogorov equation:

$$q_{t+s}(x,z) = \sum_y q_t(x,y) q_s(y,z) \mu_y = (q_t^x, q_s^z).$$

Thus $q_t(x,\cdot) \in L^2$ for each x, t.

The following relation gives one key to the control of $q_t(x,x)$. Let $x \in G$, $\psi(t) = (q_t^x, q_t^x) = q_{2t}(x,x)$. Then

$$\dot\psi(t) = 2(\tfrac{\partial}{\partial t} q_t^x, q_t^x) = 2(\Delta q_t^x, q_t^x) = -2\mathcal{E}(q_t^x, q_t^x). \tag{3.7}$$

Note that (4.7) shows that $q_t(x,x)$ is strictly decreasing in t. Differentiating a second time we have

$$\ddot\psi(t) = 4(\Delta q_t^x, \Delta q_t^x) > 0, \tag{3.8}$$

so $\dot\psi(t)$ is strictly increasing.

Theorem 3.6. *Let $\alpha \geq 1$. The following are equivalent:*
(a) Γ satisfies the Nash inequality (N_α).
(b) There exists c_H such that

$$q_t(x,y) \leq \frac{c_H}{t^{\alpha/2}}, \quad t > 0, \ x, y \in G. \tag{3.9}$$

(c) There exists c'_H such that

$$p_n(x,y) \le \frac{c'_H}{n^{\alpha/2}}, \quad n \ge 1,\ x,y \in G.$$

Proof. I will only prove the equivalence of (a) and (b). (a) $\Rightarrow$ (b). First, it is enough to prove (3.9) in the case $x = y$. For,

$$\begin{aligned} q_t(x,y) &= \sum_z q_{t/2}(x,z) q_{t/2}(y,z)\mu_z \\ &\le \Big(\sum_z q_{t/2}(x,z)^2 \mu_z\Big)^{1/2} \Big(\sum_z q_{t/2}(y,z)^2\mu_z\Big)^{1/2} = \big(q_t(x,x)q_t(y,y)\big)^{1/2}. \end{aligned}$$

Fix $x \in G$ and set

$$\psi(t) = (q_t^x, q_t^x) = q_{2t}(x,x).$$

Now $||q_t^x||_1 = 1$ and $||q_t^x||_2^2 = \psi(t) < \infty$. So $q_t^x \in L^1 \cap L^2$, and using (3.7) and (N_α)

$$\dot\psi(t) = -2\mathcal{E}(q_t^x, q_t^x) \le -c_N ||q_t^x||_2^{2+4/\alpha} ||q_t^x||_1^{-4/\alpha} = -c_N \psi(t)^{1+2/\alpha}.$$

This differential inequality for ψ now leads easily to estimates on its asymptotic behaviour. Let $\varphi(t) = \psi(t)^{-2/\alpha}$; then

$$\dot\varphi(t) = -\frac{2}{\alpha}\dot\psi(t)\psi(t)^{-1-2/\alpha} \ge c_2.$$

So $\varphi(t) \ge \varphi(0) + c_2 t \ge c_2 t$, and thus $\psi(t) \le (c_2 t)^{-\alpha/2}$. □

(b) $\Rightarrow$ (a). Suppose that $q_t(x,y) \le c_1 t^{-\alpha/2}$ for all x, y, t. Let $f \in L^1 \cap L^2$. Then

$$\begin{aligned} \|Q_t f\|_2^2 &= \sum_x \mu_x |Q_t f(x)|^2 \\ &\le \sum_x \sum_y \sum_z \mu_x (q_t(x,y)|f(y)|\mu_y)(q_t(x,z)|f(z)|\mu_z) \\ &= \sum_y \sum_z q_{2t}(y,z)|f(y)||f(z)|\mu_y\mu_z \le c_1(2t)^{-\alpha/2}\|f\|_1^2. \end{aligned} \tag{3.10}$$

Set $\psi(t) = \|Q_t f\|_2^2$, and write $\psi(0) = \|f\|_2^2$. Then as in (3.7), (3.8) we have

$$\dot\psi(t) = 2(\tfrac{\partial}{\partial t} Q_t f, Q_t f) = 2(\Delta Q_t f, Q_t f) = -2\mathcal{E}(Q_t f, Q_t f),$$

while

$$\ddot\psi(t) = 4(\Delta Q_t f, \Delta Q_t f) \ge 0.$$

Thus $\dot\psi$ is increasing, and so $\dot\psi(t) \geq \dot\psi(0) = -2\mathcal{E}(f,f)$. Hence

$$\psi(t) - \psi(0) = \int_0^t \dot\psi(s)ds \geq \int_0^t \dot\psi(0)ds = -2t\mathcal{E}(f,f).$$

Rearranging and using (3.10) gives that, for any t,

$$2\mathcal{E}(f,f) \geq \frac{\|f\|_2^2 - \psi(t)}{t} \geq \frac{\|f\|_2^2 - c_2 t^{-\alpha/2}\|f\|_1^2}{t}. \tag{3.11}$$

We could now choose t to optimise the right hand side of (3.11), but we only lose a constant if we choose t so that $c_2 t^{-\alpha/2}\|f\|_1^2 = \frac{1}{2}\|f\|_2^2$. Then (3.11) yields

$$2\mathcal{E}(f,f) \geq c\|f\|_2^2(\|f\|_2^2/\|f\|_1^2)^{2/\alpha} = c\|f\|_2^{2+4/\alpha}\|f\|_1^{-4/\alpha}.$$

□

Examples. 1. Let $\Gamma = \mathbb{Z}^d$, $\mu_{xy}^{(0)}$ be the natural weights on $\mathbb{Z}^d$, and $\mu_{xy}^{(1)}$ be weights on $\mathbb{Z}^d$ satisfying $\mu_{xy}^{(1)} \geq c_0\mu_{xy}^{(0)}$. Let $\mathcal{E}_i$ be the Dirichlet forms associated with $\Gamma, \mu^{(i)}$, and write $X^{(i)}$, $Y^{(i)}$ for the associated SRW (discrete and continuous time). By Polya's calculations we know that the SRW on $(\Gamma, \mu^{(0)})$ satisfies the bound $p_n^{(0)}(x,y) \leq cn^{-d/2}$. Using (4.4) it follows that $q_t^{(0)}(x,y) \leq ct^{-d/2}$, so by the Theorem we have that the Nash inequality (N_d) holds with respect to $\mathcal{E}_0$.

Since $\mathcal{E}_1(f,f) \geq c_0\mathcal{E}_0(f,f)$, we have also (N_d) with respect to $\mathcal{E}_1$, and therefore the bound $q_t^{(1)}(x,y) \leq c't^{-d/2}$ holds.

2. The following simple proof of the Nash inequality in $\mathbb{R}^d$ is due to Stein – see p. 935 of [N]. Let $f \in C^2(\mathbb{R}^d)$, and $\widehat{f}(\theta)$ be the Fourier transform of f. Then

$$\int |\widehat{f}(\theta)|^2 d\theta = \int |f|^2 = \|f\|_2^2,$$

while

$$\mathcal{E}(f,f) = \int |\nabla f|^2 = \int |\theta|^2 |\widehat{f}(\theta)|^2 d\theta.$$

However,

$$|\widehat{f}(\theta)| = (2\pi)^{-n/2} |\int e^{i\theta x} f(x)dx| \leq (2\pi)^{-n/2}\|f\|_1.$$

Let $r > 0$, and C_d be the volume of the unit sphere in $\mathbb{R}^d$. Then

$$\begin{aligned}
\|f\|_2^2 &= \int_{B(0,r)} |\widehat{f}(\theta)|^2 + \int_{B(0,r)^c} |\widehat{f}(\theta)|^2 \\
&\leq C_d r^d \|f\|_1^2 + \int_{B(0,r)^c} |\theta/r|^2 |\widehat{f}(\theta)|^2 \\
&\leq C_d r^d \|f\|_1^2 + r^{-2}\mathcal{E}(f,f).
\end{aligned}$$

If we choose r so that the last two terms are equal then $r^{d+2} = \mathcal{E}(f,f)/\|f\|_1^2$, and we obtain

$$\|f\|_2^2 \le C'_d \mathcal{E}(f,f)^{d/(d+2)} \|f\|_1^{4/(d+2)},$$

which is (N_d).

3. Let G be the join of two copies of $\mathbb{Z}^d$ at their origins, which we write as G_i, $i = 1, 2$. Let $f \in C_0(G)$, and write $f_i = f|_{G_i}$. Choose i such that $\|f_i\|_2^2 \ge \frac{1}{2}\|f\|_2^2$. Then using the Nash inequality in G_i,

$$\|f_i\|_2^2 \le 2\|f_i\|_2^2 \le c\mathcal{E}(f_i, f_i)^{d/(d+2)} \|f_i\|_1^{4/(d+2)} \le c\mathcal{E}(f,f)^{d/(d+2)} \|f\|_1^{4/(d+2)},$$

so G also satisfies (N_d). Thus, by Theorem 3.6 the transition density of the CTSRW on G also satisfies the bound

$$q_t(x,y) \le Ct^{-d/2}.$$

Definition. Let $x_0 \in A$. (Γ, μ) satisfies the rooted isoperimetric inequality (I_α^R) if

$$\mu_E(A; G - A) \ge c\mu(A)^{(\alpha-1)/\alpha} \text{ for all connected } A \text{ with } x_0 \in A.$$

This arises in percolation contexts – see [BLS].

Problem. Does (I_α^R) imply that

$$q_t(x_0, x_0) \le ct^{-\alpha/2}? \tag{3.12}$$

If we try to follow through the arguments above we can obtain the inequalities (S_α^1), (S_α^1) and (N_α) for functions f such that $\Lambda_t(f)$ is connected and contains x_0 – call this space of functions $\mathcal{R}$. But to prove (3.12) by the method of Theorem 3.6 we would need $q_t(x_0, \cdot) \in \mathcal{R}$, which is not true in general.

We have seen that if $\alpha > 2$ then (I_α) implies (N_α). The argument above, which goes via (S_α^2), cannot work for $\alpha \in (1, 2]$. See Coulhon [C1] for the proof of

Theorem 3.7. *(I_α) implies (N_α) for $\alpha \in (1, \infty)$.*

The survey [C1] also presents a more systematic way of viewing this (and other) families of Sobolev type inequalities.

Example. The graphical (pre)-Sierpinski carpet Γ_{SC} is an infinite connected graph (a subset of $\mathbb{Z}_+^2$) which is roughly isometric with the (unbounded) Sierpinski carpet. For a precise definition see [BBGSC].

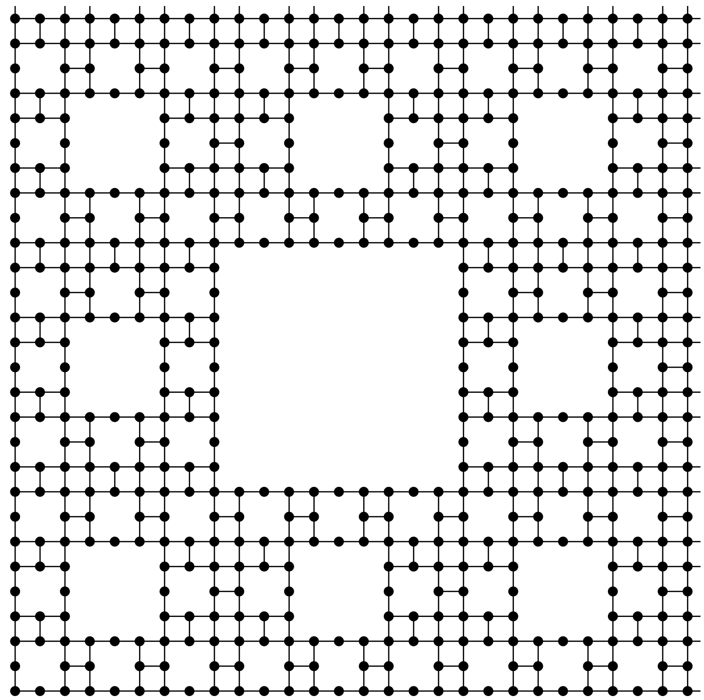

Figure 3.1: The graphical Sierpinski carpet.

Let $A_n = G \cap [0, \frac{1}{2}3^n]^2$. Then $\mu(A_n) \approx c8^n$, and $\mu_E(A_n, G - A_n) \approx 2^n$. So if G satisfies (I_α) then

$$2^n \geq c(8^n)^{(\alpha-1)/\alpha},$$

which implies that $\alpha \leq 3/2$. In fact (as is fairly clear) the sets A_n are the worst case as far as isoperimteric properties in G are concerned.

Theorem 3.8. *(Osada, [O1].) Γ_{SC} satisfies (I_α) with $\alpha = 3/2$.*

Using this and Theorems 3.6 and 3.7 it follows that the SRW on G satisfies

$$p_n(x, y) \leq cn^{-3/4}. \tag{3.13}$$

However, this is not the correct rate of decay; in [BBGSC] it is proved that

$$p_n(x, y) \asymp n^{-\gamma},$$

where $\gamma = (\log 8)/(\log 8\rho)$. Here the exact value of ρ is unknown, but it is close to $5/4$, so that $\gamma \approx 0.90 > 0.75$.

The reason (3.13) is not best possible is that there is a loss of information in passing from (I_α) or (S^1_α) to (N_α). The inequality (S^1_α) implies that (in a certain sense) the graph Γ has 'good geometry', while (N_α) or (S^2_α) imply 'good heat flow'. Good geometry is sufficient, but not necessary, for good heat flow.

4. Evolving random sets and upper bounds.

This idea is due to Morris and Peres [MP]. The account here is rather less general than that in [MP], and concentrates on the case of infinite graphs.

Let (Γ, μ) be an infinite graph, and $\mathcal{K}$ be the set of finite subsets of G. We define a $\mathcal{K}$ valued Markov Chain $(S_n, n \geq 0)$ in the following way. Let

$$\mu(x, A) = \sum_{y \in A} \mu_{yx}. \tag{4.1}$$

Let $(U_k, k \geq 1)$ be i.i.d. $U(0,1)$ r.v. Given $S_0, \dots, S_n$, let

$$S_{n+1} = \{y : \mu(y, S_n) \geq \mu_y U_{n+1}\}. \tag{4.2}$$

Notes. 1. y cannot be added to S_n unless $y \in \partial S_n$.
2. If $y \in S_n^o$ then $y \in S_{n+1}$.
3. $\emptyset$ is absorbing for (S_n).

We write P_A for the law of (S_n) starting at $S_0 = A$.

Lemma 4.1. $P_{\{x\}}(y \in S_n) = \mu_x p_n(x, y)$.

Proof. If $n = 0$ then $P_{\{x\}}(y \in S_0) = \delta_{xy}$, while $\mu_x p_0(x, y) = \mu_x \delta_{xy}/\mu_y = \delta_{xy}$. Now suppose the result holds for n, and for all x, y. Then

$$\begin{aligned}
P_{\{x\}}(y \in S_{n+1}) &= E_{\{x\}} P(U_{n+1} \leq \frac{\mu(y, S_n)}{\mu_y}) \\
&= E_{\{x\}} \mu_y^{-1} \mu(y, S_n) \\
&= E_{\{x\}} \mu_y^{-1} \sum_z 1_{(z \in S_n)} \mu_{yz} \\
&= \sum_z \mu_y^{-1} \mu_{yz} p_n(x, z) \mu_x \\
&= \mu_x \sum_z p_1(y, z) p_n(z, x) \mu_z = \mu_x p_{n+1}(x, y).
\end{aligned}$$

□

Lemma 4.2. $\mu(S_n)$ *is a martingale.*

Proof. We have

$$\begin{aligned}
E_{\{x\}}(\mu(S_{n+1}) | S_n) &= \sum_y \mu_y P_{\{x\}}(y \in S_{n+1} | S_n) \\
&= \sum_y \mu_y \frac{\mu(y, S_n)}{\mu_y} = \sum_y \mu(y, S_n) = \mu(S_n).
\end{aligned}$$

□

Lemma 4.3.

$$p_{2n}(x,x) \le \frac{(E_{\{x\}}\mu(S_n)^{1/2})^2}{\mu_x^2}. \tag{4.3}$$

Proof. Let (S_n), (S'_n) be independent copies of the ERS process. Then

$$\begin{aligned} p_{2n}(x,x) &= \sum_y p_n(x,y)p_n(y,x)\mu_y \\ &= \mu_x^{-2}\sum_y P_{\{x\}}(y\in S_n)^2\mu_y \\ &= \mu_x^{-2}\sum_y P_{\{x\}}(y\in S_n)P_{\{x\}}(y\in S'_n)\mu_y \\ &= \mu_x^{-2}E_{\{x\}}\Big(\sum_y 1_{(y\in S_n)}1_{(y\in S'_n)}\mu_y\Big) \\ &\le \mu_x^{-2}E_{\{x\}}\Big(\big(\sum_y 1_{(y\in S_n)}\mu_y\big)^{1/2}\big(\sum_y 1_{(y\in S'_n)}\mu_y\big)^{1/2}\Big) \\ &= \mu_x^{-2}E_{\{x\}}\big(\mu(S_n)^{1/2}\mu(S'_n)^{1/2}\big) = \mu_x^{-2}(E_{\{x\}}\mu(S_n)^{1/2})^2. \end{aligned}$$

These three results are quite general, and use almost nothing. (They also hold for non-reversible chains, but I will not go into that.)

We now make two assumptions on (Γ,μ) :

(1) (Γ,μ) satisfies (I_α).

(2) The random walk on (Γ,μ) is lazy: that is $\mathbb{P}^x(X_1 = x) \ge \frac{1}{2}$. Note this implies that $\mu_{xx} \ge \frac{1}{2}\mu_x$ for $x \in G$.

We now use Lemma 4.3 to bound $p_{2n}(x,x)$. Since $\mu(S_n)$ is a non-negative martingale, it converges as $n \to \infty$. If $\mu(S_n)$ is large, then (I_α) implies that $\mu(S_n, S_n^c)$ is also large, so that $\mu(S_{n+1}) - \mu(S_n)$ should also be large on average. This should then prevent $\mu(S_n)$ from settling down at some non-zero value, and lead to some control on the speed at which $\mu(S_n) \to 0$.

We now implement this intuition. As we will see the argument uses a number of clever tricks.

For $A \subset G$ (with $A \ne \emptyset$) set

$$\gamma(A) = \frac{\mu(A,A^c)}{\mu(A)}.$$

The lazy condition implies that $\gamma(A) \le \frac{1}{2}$, and (I_α) gives

$$\gamma(A) \ge c\mu(A)^{-1/\alpha}. \tag{4.4}$$

Let $S_0 = A$ and suppose $U_1 < \frac{1}{2}$. If $y \in A$ then $\mu(y,A) \geq \mu_{yy} \geq \frac{1}{2}\mu_y > U_1\mu_y$, so $y \in S_1$. If $y \in \partial A$ then $y \in S_1$ if and only if $U_1 < \mu(y,A)/\mu_y$. So,

$$P_A(y \in S_1 | U_1 < \tfrac{1}{2}) = \begin{cases} 1 & \text{if } y \in A, \\ 2\mu(y,A)/\mu_y & \text{if } y \in \partial A. \end{cases}$$

Hence

$$E_A(\mu(S_1)|U_1 < \tfrac{1}{2}) = \mu(A) + 2\mu(A,A^c) = \mu(A)(1+2\gamma(A)),$$

and as $\mu(S_n)$ is a martingale

$$E_A(\mu(S_1)|U_1 > \tfrac{1}{2}) = \mu(A) - 2\mu(A,A^c) = \mu(A)(1-2\gamma(A)).$$

Now it is easy to check, by squaring, that for $|t| \leq \frac{1}{2}$,

$$\tfrac{1}{2}(1+2t)^{1/2} + \tfrac{1}{2}(1-2t)^{1/2} \leq\leq (1-t^2)^{1/2} \leq 1 - \tfrac{1}{2}t^2.$$

So, using Jensen,

$$\begin{aligned} E_A(\mu(S_1)^{1/2}) &= \tfrac{1}{2}E_A(\mu(S_1)^{1/2}|U_1 < \tfrac{1}{2}) + \tfrac{1}{2}E_A(\mu(S_1)^{1/2}|U_1 > \tfrac{1}{2}) \\ &\leq \tfrac{1}{2}E_A(\mu(S_1)|U_1 < \tfrac{1}{2})^{1/2} + \tfrac{1}{2}E_A(\mu(S_1)|U_1 > \tfrac{1}{2})^{1/2} \\ &= \tfrac{1}{2}(\mu(A)(1+2\gamma(A)))^{1/2} + \tfrac{1}{2}(\mu(A)(1-2\gamma(A)))^{1/2} \\ &\leq \mu(A)^{1/2}(1-\gamma(A)^2/2). \end{aligned}$$

Thus by (4.4), if $A \neq \emptyset$,

$$E_A(\mu(S_1)^{1/2}) \leq \mu(A)^{1/2}(1 - c\mu(A)^{-2/\alpha}). \tag{4.5}$$

Lemma 4.4. *Let M_n be a non-negative martingale, and $\delta \geq 0$. Then*

$$E(M_n^{\frac{1}{2}} 1_{(M_n>0)})^{1-\delta} \geq (EM_n^{1/2})^{1+\delta}.$$

Proof. Let η, ξ be non-negative r.v. with $E\eta = 1$. Then $\hat{E}(\xi) = E\eta\xi$ defines a new probability $\hat{P}$, and applying Jensen

$$E\eta\xi^{1+p} = \hat{E}\xi^{1+p} \geq (\hat{E}\xi)^{1+p} = (E\eta\xi)^{1+p}.$$

Then

$$\begin{aligned} EM_n^{\frac{1}{2}-\frac{1}{2}\delta} 1_{(M_n>0)} &= EM_nM_n^{-\frac{1}{2}(1-\delta)}1_{(M_n>0)} \\ &\geq (EM_nM_n^{-1/2}1_{(M_n>0)})^{1+\delta} = (EM_n^{1/2})^{1+\delta}. \end{aligned}$$

□

Now let $Y_n = \mu(S_n)^{1/2}$, and $y_n = E_{\{x_0\}}(Y_n)$. Then (4.5) gives

$$E_{\{x_0\}}(Y_{n+1}|Y_n) \le Y_n - cY_n^{1-4/\alpha} 1_{(Y_n>0)}. \tag{4.6}$$

So by Lemma 4.4,

$$y_{n+1} - y_n \le -cEY_n^{1-4/\alpha} 1_{(Y_n>0)} = -cEM_n^{\frac{1}{2}(1-4/\alpha)} 1_{(M_n>0)} \le -cy_n^{1+4/\alpha}. \tag{4.7}$$

This (apart from the power) is the discrete analogue of the differential inequality $\varphi'(t) \le -c\varphi(t)^{1+2/\alpha}$ that arose in Theorem 3.6. Let $\delta = 4/\alpha$; we have

$$y_{n+1} \le y_n(1 - cy_n^\delta) \le y_n \exp(-cy_n^\delta).$$

Then

$$\int_{y_{n+1}}^{y_n} t^{-1-\delta} dt \ge y_n^{-\delta} \log(y_n/y_{n+1}) \ge c,$$

so

$$n \le \int_{y_n}^{y_0} t^{-1-\delta} dt \le cy_n^{-\delta},$$

giving

$$y_n \le cn^{-\alpha/4}, \qquad n \ge 1.$$

By Lemma 4.3 we obtain

Theorem 4.5. *Suppose (Γ, μ) satisfies (I_α), and random walk on (Γ, μ) is lazy. Then*

$$p_{2n}(x_0, x_0) \le cn^{-\alpha/2}, \qquad n \ge 1.$$

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